MTHSc 853: Linear Algebra

2. Duality.

Let X be a vector space over a field K. Recall that a scalar function $\ell : X \longrightarrow K$ is linear if

$$\ell(x+y) = \ell(x) + \ell(y) \quad \text{and} \quad \ell(ax) = a\,\ell(x) \quad \forall\, x, y \in X,\ a \in K,$$

or equivalently, if $\ell(a_1x_1 + \cdots + a_nx_n) = a_1\ell(x_1) + \cdots + a_n\ell(x_n)$,

for all $a_i \in K$, $x_i \in X$.

The set of linear functions $X \to K$ itself forms a linear space, called the dual of X, denoted X'.

Addition and scalar multiplication is defined naturally:

* Addition: $(\ell + m)(x) = \ell(x) + m(x)$
* Scalar mult: $(a\ell)(x) = a\,\ell(x)$.

Example 1: Let $X = \{\text{continuous functions } [0,1] \to \mathbb{R}\}$.

Let $s_1, \ldots, s_n \in [0,1]$. Then the following are all linear functions:

(i) $\ell(f) = f(s_1)$

(ii) $\ell(f) = \sum_{i=1}^{n} a_i f(s_i)$, $a_i \in \mathbb{R}$.

(iii) $\ell(f) = \int_0^1 f(s)\,ds$.

Example 2: Let $X = \{\text{n-differentiable functions } [0,1] \to \mathbb{R}\}$. For $s \in [0,1]$,

$\ell(f) = \sum_{i=1}^{n} a_i \partial^i f(s)$ is a linear function.

<u>Theorem 2.1</u>: Let X be a vector space of dimension $n<\infty$. Write elements of X as n-tuples $x=(c_1,\dots,c_n)$, $c_i \in K$, where addition and scalar multiplication is componentwise. Fix $a_1,\dots,a_n \in K$, and the function $\ell(x) = a_1c_1 + \dots + a_nc_n$ is linear.

Conversely, every linear function $\ell : X \to K$ can be expressed like this.

<u>Proof</u>: It is obvious that $\ell(x) = a_1c_1+\dots+a_nc_n$ is linear.

For the converse, let $\ell: X \to K$ be linear. Let $e_i = (0,\dots,0,\overset{\text{pos. } i}{1},0,\dots,0)$

Then $x=(c_1,\dots,c_n)$ can be expressed as $x = c_1e_1+\dots+c_ne_n$.

Put $a_i = \ell(e_i)$ for each $i=1,\dots,n$. This works. (Check!) $\square$

Thus, we can think of vectors $x \in X$ as n-tuples of scalars, and linear maps $\ell \in X'$ as n-tuples of scalars as well. In both cases, addition and scalar multiplication works componentwise. Thus, we deduce the following:

<u>Theorem 2.2</u>: The dual X' of a finite-dimensional vector space X is a finite dimensional vector space, and $\dim X' = \dim X$.

A linear function ℓ applied to a vector $x \in X$ depends on the n-tuples $(c_1,\dots,c_n)$ for x, and $(a_1,\dots,a_n)$ for ℓ. We use a <u>scalar product</u> notation $(\ell, x) := \ell(x)$.

Remark: If $\dim X = n < \infty$, then $X \simeq K^n$ (exercise), and so by Theorem 2.2, $X \simeq X'$.

This fails if $\dim X = \infty$.

Example: Let $\ell^1 = \{(a_1, a_2, a_3, \dots) : a_i \in \mathbb{R}, \sum_{i=1}^{\infty} |a_i| < \infty\}$.

If $x, y \in \ell^1$, with $x = (a_1, a_2, \dots)$, $y = (b_1, b_2, \dots)$,
then $(x, y) := \sum_{i=1}^{\infty} a_i b_i < \infty$, so every $y \in \ell^1$ defines a linear function $\ell^1 \to \mathbb{R}$, i.e., $\ell^1 \subseteq (\ell^1)'$ (under the natural identification of ℓ^1 with elements in $(\ell^1)'$.)

But there are others!

Note that if $z = (1, 1, 1, \dots)$ then $(x, z) = a_1 + a_2 + a_3 + \dots < \infty$.

But $z \notin \ell^1$.

* The scalar product is a bilinear function of ℓ and x. That is, for fixed ℓ (resp. x) it is a linear function of x (resp. ℓ)

Equivalently, $(a\ell, x) = \underbrace{a(\ell, x)}_{a\,\ell(x)} = \underbrace{(\ell, ax)}_{\ell(ax)}$ for all $x \in X$, $\ell \in X'$, $a \in K$.

Since X' is a vector space, it has a dual X'' consisting of all linear functions on X'.

For a <u>fixed</u> $x \in X$, the scalar product is such a linear function.

By Theorem 2.1, <u>all</u> linear functions $X' \to K$ are of this form.

<u>Theorem 2.3</u>: The bilinear (l, x) gives a natural identification (isomorphism) of X with X''.

Explicitly, this is:

$$\begin{array}{ccc} X & \longrightarrow & X'' = \{X' \to K\} \\ x & \longmapsto & \left(\begin{array}{ccc} X' & \longrightarrow & K \\ l & \xmapsto{(l,x)} & l(x) \end{array} \right) \end{array}$$

<u>Def</u>: Let Y be a subspace of X. The set of linear functions l that vanish on Y, i.e., $\{l \in X' : l(y) = 0 \ \forall y \in Y\}$ is the <u>annihilator</u> of Y, denoted $Y^\perp$ (or Y° or $\operatorname{ann}(Y)$).

It's easily verified that $Y^\perp$ is a subspace of X'.

<u>Theorem 2.4</u>: Let Y be a subspace of a finite dimensional space X. Then $Y^\perp + \dim Y = \dim X$.

Proof: We will construct a natural isomorphism $\Phi: Y^\perp \longrightarrow (X/Y)'$

$$\ell \overset{\Phi}{\longmapsto} L$$

Define the map $L\{x\} = \ell(x)$, that is, $L: X/Y \longrightarrow K$

$$\{x\} \longmapsto \ell(x)$$

Need to check this is <u>well-defined</u>, i.e, that it does not depend on the choice of $x \in \{x\}$.

Indeed: If $x \equiv x' \bmod Y$, then $x' = x + y$ for some $y \in Y$

$$\Rightarrow \ell(x') = \ell(x+y) = \ell(x) + \ell(y) = \ell(x) \checkmark$$

Also note that L is <u>linear</u> and <u>injective</u> (both easy.)

Conversely, <u>any</u> such L in $(X/Y)'$ defines a linear function $\ell \in X'$ satisfying $\ell(y) = 0 \ \forall y \in Y$.

Thus, $\Phi: \ell \mapsto L$ is a bijection & defines an isomorphism, so

$\dim Y^\perp = \dim (X/Y)'$ (*)

We now have $\dim X = \dim Y + \dim (X/Y)$ Thm 1.6

$= \dim Y + \dim (X/Y)'$ Thm 2.2

$= \dim Y + \dim Y^\perp$ (*) □

<u>Def</u>: The dimension of $Y^\perp$ is the <u>codimension</u> of Y in X, denoted codim Y.

By Theorem 2.4, $\operatorname{codim} Y + \dim Y = \dim X$.

* Since $Y^\perp$ is a subspace of X', its annihilator, denoted $Y^{\perp\perp}$ is a subspace of X''.

<u>Theorem 2.5</u>: Let Y be a subspace of X, finite-dimensional. Identifying X with X'' (henceforth), we have $Y^{\perp\perp} = Y$.

<u>Proof</u>: Pick $y \in Y$, consider $l \in Y^\perp$.

By def'n, $(l, y) = l(y) = 0 \Rightarrow Y \subseteq Y^{\perp\perp}$. (under $Y \leftrightarrow Y''$)

To show $Y = Y^{\perp\perp}$, it suffices to show $\dim Y = \dim Y^{\perp\perp}$

We have
$$\begin{aligned} \dim Y^{\perp\perp} + \dim Y^\perp &= \dim X' && \text{Thm 2.4} \\ &= \dim X && \text{Thm 2.2} \\ &= \dim Y^\perp + \dim Y && \text{Thm 2.4.} \end{aligned}$$
$\square$

<u>Remark</u>: We can also define the <u>annihilator</u> of an arbitrary subset $S \subseteq X$ to be $S^\perp = \{ l \in X' : l(s) = 0 \ \forall s \in S \}$.

<u>Theorem 2.6</u>: Let $S \subseteq X$; finite dim'l. Let Y be the <u>smallest</u> subspace containing S $\left(= \operatorname{Span}(S), \ \underline{\text{or}} = \bigcap_{\substack{\text{subspaces } T \\ T \supseteq S}} T \right)$.

Then $S^\perp = Y^\perp$

<u>Proof</u>: Exercise.