*Read*: Lax, Chapter 1, pages 1–11.

1. Let $X$ be a vector space over a field $K$. Let 0 be the zero element of $K$ and $\mathbf{0}$ the zero-element of $X$. Using only the definitions of a group, a vector space, and a field, carefully prove each of the following:
   (a) The identity element $e$ of a group is unique.
   (b) In any group $G$, the inverse of $g \in G$ is unique.
   (c) $0x = \mathbf{0}$ for every $x \in X$;
   (d) $k\mathbf{0} = \mathbf{0}$ for every $k \in K$;
   (e) For every $k \in K$ and $x \in X$, if $kx = \mathbf{0}$, then $k = 0$ or $x = \mathbf{0}$.

2. The following is called the *Replacement Lemma*: Let $X$ be a vector space over $K$, and let $S$ be a linearly independent subset of $X$. Let $x_0 \in \mathrm{Span}(S)$ with $x_0 \neq 0$. Then there exists $x_1 \in S$ such that the set $S' = (S \setminus \{x_1\}) \cup \{x_0\}$ is a basis for $\mathrm{Span}(S)$.
   (a) Prove the Replacement Lemma.
   (b) Suppose that $B$ is a basis for $X$ containing $n$ elements, and let $B'$ be another basis for $X$. Use the Replacement Lemma to show that $|B'| = n$.

3. Let $S$ be a set of vectors in a finite-dimensional vector space $X$. Show that $S$ is a basis of $X$ if every vector of $X$ can be written in one and only one way as a linear combination of the vectors in $S$.

4. Let $X_1$, $X_2$ be a subspace of a finite-dimensional $K$-vector space $X$. Show that $\dim(X_1 \times X_2) = \dim(X_1 \oplus X_2)$.

5. If $Y$ is a subspace of $X$, then two vectors $x_1, x_2 \in X$ are *congruent modulo* $Y$, denoted $x_1 \equiv x_2 \bmod Y$, if $x_1 - x_2 \in Y$. This is an equivalence relation; denote the equivalence class containing $x \in X$ by $\{x\}$, and let $X/Y$ denote the set of equivalence classes. We can make $X/Y$ into a vector space by defining addition and scalar multiplication as follows:
$$\{x\} + \{z\} = \{x + z\}, \qquad \{ax\} = a\{x\}.$$
   (a) Show that these operations are well-defined. That is, they do not depend on the choice of congruence class representatives.
   (b) Assume $\dim X < \infty$. Show that $X$ is isomorphic to $Y \times X/Y$ by defining an explicit map and showing that it is linear and a bijection.

6. Let $Y$ be a subspace of $X$. Prove that for any $x \in X$, the following two sets are equal as subsets of $X$:
$$\{x\} = \{x + y \mid y \in Y\}.$$
   This motivates the alternative "coset notation" of $x + Y$ for the equivalence class of $x$ modulo $Y$. Show how to add and scalar multiply in this notation by computing:
$$(x + Y) + (z + Y), \qquad \text{and} \qquad a(x + Y), \quad x, z \in X,\ a \in K.$$