1. Let K be a finite field. The *characteristic* of K, denoted $\operatorname{char} K$, is the smallest positive integer n for which $n1 := \underbrace{1 + 1 + \cdots + 1}_{n \text{ times}} = 0$.

   (a) Prove that the characteristic of K is prime.

   (b) Show that K is a vector space over $\mathbb{Z}_p$, where $p = \operatorname{char} K$.

   (c) Show that the order $|K|$ of K (the number of elements it contains) is a prime power.

   (d) Show that if K and L are finite fields with $K \subset L$ and $|K| = p^m$ and $|L| = p^n$, then m divides n.

2. Let X be a vector space over a field K and let X' be the the set of linear functions from X to K, also known as the *dual space* of X.

   (a) Let $v_1, \dots, v_n$ be a basis for X. For each i, show there exists a unique linear map $f_i \colon X \to K$ such that $f_i(v_i) = 1$ and $f_i(v_j) = 0$ for $j \neq i$.

   (b) Show that $f_1, \dots, f_n$ is a basis for X' (called the *dual basis* of $v_1, \dots, v_n$).

   (c) Consider the basis $v_1 = (1, -1, 3)$, $v_2 = (0, 1, -1)$, and $v_3 = (0, 3, -2)$ of $X = \mathbb{R}^3$. Find a formula for each element of the dual basis.

   (d) Express the linear map $f \in X'$, where $f(x, y, z) = 2x - y + 3z$ as a linear combination of the dual basis, f_1, f_2, f_3.

3. Let S be a subset of X. The *annihilator* of S is the set

$$S^\perp = \{\ell \in X' \mid \ell(s) = 0 \ \text{ for all } s \in S\}.$$

   (a) Show that $S^\perp$ is a subspace of X'.

   (b) Show that $\operatorname{span}(S)$ is the intersection of all subspaces T_α of X that contain S:

$$\operatorname{span}(S) = \bigcap_{S \subseteq T_\alpha \leq X} T_\alpha,$$

   making it well-founded to speak of the "*smallest subpace of X that contains S.*"

   (c) Let $Y = \operatorname{span}(S)$. Show that $S^\perp = Y^\perp$.

4. Let $\mathcal{P}_2$ be the vector space of all polynomials $p(x) = a_0 + a_1 x + a_2 x^2$ over $\mathbb{R}$, with degree ≤ 2. Let ξ_1, ξ_2, ξ_3 be distinct real numbers, and define

$$\ell_j \colon \mathcal{P}_2 \longrightarrow \mathbb{R}, \qquad \ell_j(p) = p(\xi_j) \quad \text{for} \quad j = 1, 2, 3.$$

   (a) Show that ℓ_1, ℓ_2, ℓ_3 is a basis for the dual space $\mathcal{P}_2'$.

   (b) Find polynomials $p_1(x), p_2(x), p_3(x)$ in $\mathcal{P}_2$ of which ℓ_1, ℓ_2, ℓ_3 is the dual basis in $\mathcal{P}_2'$.

5. Let W be the subspace of $\mathbb{R}^4$ spanned by $(1, 0, -1, 2)$ and $(2, 3, 1, 1)$. Which linear functions $\ell(x) = c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4$ are in the annihilator of W? Write your answer by giving an explicit basis of $W^\perp$.