1. Let $T\colon X \to U$ be a linear map. Prove the following:

   (a) The image of a subspace of X is a subspace of U.

   (b) The inverse image of a subspace of U is a subspace of X.

2. Let X and U be vector spaces, and suppose that Y is a subspace of X. Let $Q\colon X \to X/Y$ be the canonical quotient map sending $x \xmapsto{Q} \{x\}$, and let $T\colon X \to U$ be a linear map. Give necessary and sufficient conditions for the existence of a unique linear map $S\colon X/Y \to U$ such that $T = S \circ Q$. When this happens, the map T is said to *factor through* the quotient space, as shown by the following commutative diagram:

$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[dr, "Q"'] & & U \\ & X/Y \arrow[ur, dashed, "S"'] & \end{tikzcd}$$

   Prove all of your claims.

3. Suppose $T\colon X \to X$ is a linear map of rank 1.

   (a) Show that there exists $c \in K$ such that $T^2 = cT$.

   (b) Show that if $c \neq 1$, then $I - T$ has an inverse.

4. Suppose that $S, T\colon X \to X$ are linear maps.

   (a) Show that $\operatorname{rank}(S+T) \leq \operatorname{rank}(S) + \operatorname{rank}(T)$.

   (b) Show that $\operatorname{rank}(ST) \leq \operatorname{rank}(S)$.

   (c) Show that $\dim(N_{ST}) \leq \dim N_S + \dim N_T$.

   For each of these, give an explicit example showing how equality need not hold.

5. Let $A, B\colon X \to X$ be linear maps.

   (a) Show that if A is invertible and similar to B, then B is also invertible, and B^{-1} is similar to A^{-1}.

   (b) Show that if either A or B is invertible, then AB and BA are similar.

6. Let $T : X \to X$ be linear, with $\dim X = n$.

   (a) Prove that if $T^2 = T$, then $X = R_T \oplus N_T$.

   (b) Show by example that if $T^2 \neq T$, then $X = R_T \oplus N_T$ need not hold.

   (c) Prove that $N_{T^n} = N_{T^{n+1}}$ and $R_{T^n} = R_{T^{n+1}}$.

   (d) Prove that $X = R_{T^n} \oplus N_{T^n}$.

   (e) Show there exists a linear map $S : X \to X$ such that $ST = TS$ and $ST^{n+1} = T^n$.