Throughout, X is assumed to be a vector space of dimension $n < \infty$.

1. Show that whenever meaningful,

   (a) $(ST)' = T'S'$

   (b) $(T+R)' = T' + R'$

   (c) $(T^{-1})' = (T')^{-1}$.

   Here, S' denotes the transpose of S. Carefully describe what you mean by "whenever meaningful" in each case.

2. Give a direct algebraic proof of $N_{T'}^{\perp} = (R_T^{\perp})^{\perp}$. (You may use the fact that $N_{T'} = R_T^{\perp}$, but don't simply take the annihilator of both sides of this equation.)

3. Let $\mathcal{P}_n$ be the vector space of all polynomials over $\mathbb{R}$ of degree less than n.

   (a) Show that the map $T\colon \mathcal{P}_3 \to \mathcal{P}_4$ given by
   $$T(p(x)) = 6\int_1^x p(t)\,dt$$
   is linear. Indicate whether it is 1–1 or onto.

   (b) Let $\mathcal{B}_3 = \{1, x, x^2\}$ be a basis for $\mathcal{P}_3$ and let $\mathcal{B}_4 = \{1, x, x^2, x^3\}$ be a basis for $\mathcal{P}_4$. Find the matrix representation of T with respect to these bases.

4. Let $T\colon X \to U$, with $\dim X = n$ and $\dim U = m$. Show that there exist bases $\mathcal{B}$ for X and $\mathcal{B}'$ for U such that the matrix of T in block form is
   $$M = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}$$
   where I_k is the $k \times k$ identity matrix, and the other blocks are either empty or contain all zeros.

5. Consider the linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ with matrix representation $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & -2 \\ -3 & 0 & 3 \end{bmatrix}$ with respect to the standard basis. What is the matrix representation of T with respect to the basis $\{(1,-1,0), (0,1,-1), (1,0,1)\}$?