Read: Lax, Chapter 5, pages 44–57.

1. Let S_n denote the set of all permutations of $\{1, \dots, n\}$.
   (a) Prove that $\operatorname{sgn}(\pi_1 \circ \pi_2) = \operatorname{sgn}(\pi_1)\operatorname{sgn}(\pi_2)$.
   (b) Prove that $\operatorname{sgn}(\tau) = -1$ for all transpositions $\tau \in S_n$.
   (c) Let $\pi \in S_n$, and suppose that $\pi = \tau_k \circ \cdots \circ \tau_1 = \sigma_\ell \circ \cdots \circ \sigma_1$, where $\tau_i, \sigma_j \in S_n$ are transpositions. Prove that $k \equiv \ell \bmod 2$.

2. Let f be a bilinear form over a K-vector space X with basis $\{x_1, x_2\}$.
   (a) Assume f is alternating. Determine a formula for $f(u, v)$ in terms of each $f(x_i, x_j)$ and the coefficients used to express u and v with this basis. [Pun intented!]
   (b) Repeat Part (a) but assume that f is symmetric and $f(x, x) = 0$ for all $x \in X$.

3. Let X be an n-dimensional vector space over a field K.
   (a) Prove that if $\operatorname{char} K \neq 2$, then every skew-symmetric multilinear form is alternating.
   (b) Give an example of a non-alternating skew-symmetric mulitlinear form.
   (c) Give an example of a non-zero alternating multilinear form such that $f(x_1, \dots, x_k) = 0$ for some set of linearly independent vectors $x_1, \dots, x_k$.

4. Let X be an n-dimensional vector space over $\mathbb{R}$, and let f be a non-degenerate symmetric bilinear form. That is, it has the additional property that for all nonzero $x \in X$, there is some $y \in X$ for which $f(x, y) \neq 0$.
   (a) Prove that the map $L \colon X \to X'$ given by $L \colon x \mapsto f(x, -)$ is an isomorphism.
   (b) Show that, given any basis $x_1, \dots, x_n$ for X, there exists a basis $y_1, \dots, y_n$ such that $f(x_i, y_j) = \delta_{ij}$.
   (c) Conversely, prove that if $\mathcal{B}_X = \{x_1, \dots, x_n\}$ and $\mathcal{B}_Y = \{y_1, \dots, y_n\}$ are sets of vectors in X with $f(x_i, y_j) = \delta_{ij}$, then $\mathcal{B}_X$ and $\mathcal{B}_Y$ are bases for X.

5. Let X be an n-dimensional vector space over $\mathbb{R}$, and let f be a non-degenerate symmetric bilinear form.
   (a) Show that there exists $x_1 \in X$ with $f(x_1, x_1) \neq 0$.
   (b) Any fixed $x_1 \in X$ for which $f(x_1, x_1) \neq 0$ induces a *linear* map $T = f(x_1, -)$. Find the dimension of the nullspace $Z_1 := N_T$, and show that the restriction of f to $Z_1 \times Z_1$ is again non-degenerate.
   (c) Prove that X has a basis $\{z_1, \dots, z_n\}$ such that $f(z_i, z_j) = \delta_{ij}$.

6. Let $A = (c_1, \dots, c_n)$ be an $n \times n$ matrix (c_i is a column vector), and let B be the matrix obtained from A by adding k times the i^{th} column of A to the j^{th} column of A, for $i \neq j$. Prove that $\det A = \det B$. You may assume that the determinant is an alternating n-linear form.