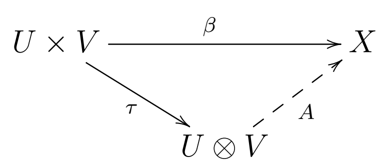
Read: Lax, Chapter 5, pages 55–56, and Appendix 4, pages 313–316.

1. Prove the following properties of the trace function:
   (a) $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for all $m \times n$ matrices A and $n \times m$ matrices B.
   (b) $\mathrm{tr}(AA^T) = \sum a_{ij}^2$ for all $n \times n$ matrices A. This quantity is the square of the *Hilbert-Schmidt norm* of A.

2. Let U, V, and X be vector spaces over a field K. Define a map

$$\tau \colon U \times V \longrightarrow U \otimes V\,, \qquad \tau(u, v) = u \otimes v\,.$$

   (a) Prove that τ is bilinear.
   (b) Prove that for any linear map $A \colon U \otimes V \to X$, the mapping $\alpha := A \circ \tau$ is bilinear from $U \times V$ to X.
   (c) Prove that for any bilinear map $\beta \colon U \times V \to X$, there is a unique linear mapping $A \colon U \otimes V \to X$ such that $\beta = A \circ \tau$. This is the *universal property* of the tensor product, and it says that any bilinear map can be "factored through" it. This is illustrated by the following commutative diagram:



3. If $\{u_1, \dots, u_n\}$ and $\{v_1, \dots, v_m\}$ are bases for U and V, respectively, then it is elementary to show that the pure tensors $\{u_i \otimes v_j \mid 1 \le i \le n,\ 1 \le j \le m\}$ span $U \otimes V$. Show that these are linearly independent, and conclude that $\dim(U \otimes V) = (\dim U)(\dim V)$. [*Hint*: Use the canonical basis $\{f_{ij}\}$ of the space of bilinear functions $U \times V \to K$, and use the universal property.]

4. Use the universal property of the tensor product to prove the following results:
   (a) $U \otimes V \cong V \otimes U$ (hint: let $X = V \otimes U$);
   (b) $(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$;
   (c) $(U \times V) \otimes W \cong (U \otimes W) \times (V \otimes W)$.