Read: Lax, Appendix 15, pages 363–366.

1. Let $X \subset \mathbb{R}[x]$ be the space of polynomials of degree $< n$ and consider the linear map

$$D : X \longrightarrow X, \qquad f \longmapsto \frac{df}{dx}.$$

Find the eigenvalues of D, and then find a basis $f_0, \dots, f_{n-1}$ of X consisting of generalized eigenvectors of D so that the matrix J with respect to this basis is in Jordan canonical form. Write down J.

2. Let A be a 7×7 matrix over $\mathbb{C}$ with minimal polynomial $m(t) = (t-1)^3(t-2)^2$.

   (a) List all possible Jordan canonical forms of A of to similarity.

   (b) For each matrix from Part (a), find the rank of $(A-I)^k$ and $(A-2I)^k$, for $k \in \mathbb{N}$.

3. Let A be an $n \times n$ matrix over $\mathbb{C}$. The matrix A is *nilpotent* if $A^k = 0$ for some $k \in \mathbb{N}$.

   (a) Prove that if A is nilpotent, then $A^n = 0$.

   (b) Prove that if A is nilpotent, then there is some $r \in \mathbb{N}$ and positive integers $k_1 \geq \cdots \geq k_r$ with $k_1 + \cdots + k_r = n$ that determine A up to similarity.

   (c) Suppose A and B are 6×6 nilpotent matrices with the same minimal polynomial and $\dim N_A = \dim N_B$. Prove that A and B are similar. Show by example that this can fail for 7×7 matrices.

4. Let A and B be $n \times n$ matrices over $\mathbb{C}$. The matrix A is *idempotent* if $A^2 = A$.

   (a) Prove that if $A^k = A$ for some integer $k > 1$, then A is diagonalizable.

   (b) Prove that idempotent matrices are similar if and only if they have the same trace.

   (c) Prove that if A and B are idempotent and $B = UAV$ holds for some invertible maps $U, V \colon X \to X$, then A and B are similar.

5. Let X be an n-dimensional vector space over $\mathbb{C}$, and let $A, B \colon X \to X$ be linear maps.

   (a) Prove that if $AB = BA$, then for any eigenvector v of A with eigenvalue λ, the vector Bv is an eigenvector of A for λ.

   (b) Show that if $\{A_1, \dots, A_k \mid A_i \colon X \to X\}$ is a set of pairwise commuting maps, then there is a nonzero $x \in X$ that is an eigenvector of every A_i.

   (c) Suppose that A and B are both diagonalizable. Prove that $AB = BA$ if and only if they are *simultaneously diagonalizable*, i.e., there exists an invertible $n \times n$-matrix P such that both $P^{-1}AP$ and $P^{-1}BP$ are diagonal matrices.