Read: Lax, Chapter 7, pages 77–100.

1. This problem is about *rational canonical form.* Consider the following matrix over $\mathbb{R}$:

$$M = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}$$

   (a) Let $e_1, \dots, e_n$ be the standard basis. Show that if a polynomial $f \in \mathbb{R}[x]$ has degree less than n, then $f(M) \neq 0$. [*Hint*: Notice that $Me_i = e_{i+1}$ for $i = 1, \dots, n-1$.]

   (b) Show that the minimal polynomial of M is

$$f(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0 \,.$$

   (c) Let X be a vector space over $\mathbb{R}$ with basis $\{x_1, x_2, x_3, x_4\}$ and let $T : X \to X$ be a linear map such that

$$T(x_1) = x_2, \quad T(x_2) = x_3, \quad T(x_3) = x_4, \quad T(x_4) = -x_1 - 4x_2 - 6x_3 - 4x_4.$$

   Find the rational and Jordan canonical forms of T. Is T diagonalizable over $\mathbb{C}$? Why or why not?

2. Given a linear map $A \colon X \to X$, define $f \colon X \to X$ by $f(x, y) = x^T Ay$.

   (a) State and prove necessary and suffcient conditions on A for f to be an inner product.

   (b) Write the inner product $f(x, y) = 3x_1y_1 - x_1y_2 - x_2y_1 + 2x_2y_2 - x_2y_3 - x_3y_2 + 3x_3y_3$ as $f(x, y) = x^T Ay$.

   (c) Find an orthonormal basis v_1, v_2, v_3 of $\mathbb{R}^3$ so that with respect to this basis, $f(z, w) = z^T Dw$ for some diagonal matrix D.

   (d) Write a formula for $f(z, w)$ like in Part (b), but with respect to this new basis.

3. Let f and g be continuous functions on the interval $[0, 1]$. Prove the following inequalities.

   (a) $\displaystyle\left(\int_0^1 f(t)g(t)\,dt\right)^2 \leq \int_0^1 f(t)^2\,dt \int_0^1 g(t)^2\,dt$

   (b) $\displaystyle\left(\int_0^1 (f(t) + g(t))^2\,dt\right)^{1/2} \leq \left(\int_0^1 f(t)^2\,dt\right)^{1/2} + \left(\int_0^1 g(t)^2 dt\right)^{1/2}.$

4. Prove that $||x|| = \sup\big\{(x, y) : y \in K^n \text{ with } ||y|| = 1\big\}$.