Read: Lax, Chapter 7, pages 89–100.

1. Use the Gram-Schmidt process to find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by $y_1 = (1, 2, 1, 1)$, $y_2 = (1, -1, 0, 2)$ and $y_3 = (2, 0, 1, 1)$.

2. Let X be the vector space of all continuous real-valued functions on $[0, 1]$. Define an inner product on X by
$$(f, g) = \int_0^1 f(t)g(t)\,dt\,.$$
Let Y be the subspace of X spanned by f_0, f_1, f_2, f_3, where $f_k(x) = x^k$. Find an orthonormal basis for Y.

3. Let Y be a subspace of an inner product space X, and $P_Y \colon X \to X$ the orthogonal projection onto Y. Prove that $P_Y^* = P_Y$.

4. Let X be a finite-dimensional real inner product space. We say that a sequence $\{A_n\}$ of linear maps converges to a limit A if $\lim_{n\to\infty} ||A_n - A|| = 0$.
   (a) Show that $\{A_n\}$ converges to A if and only if for all $x \in X$, $A_n x$ converges to Ax.
   (b) Show by example that this fails if $\dim X = \infty$.

5. Let $A \colon X \to U$ be a linear map between finite-dimensional inner product spaces, and let $A^* \colon U \to X$ denote the adjoint map. The map A has a *left inverse* if there is a linear map $L \colon U \to X$ such that $LA = I_X$, the identity on X. It has a *right inverse* if there is a linear map $R \colon U \to X$ such that $AR = I_U$ is the identity on U.
   (a) Prove that $R_{A^*}^\perp = N_A$.
   (b) Prove that A maps R_{A^*} bijectively onto R_A.
   (c) Show that if A has a left inverse, then $Ax = u$ has *at most* one solution. Give a condition on u that completely characterizes when there is a solution.
   (d) Show that if A has a right inverse, then $Ax = u$ has *at least* one solution. If $Ax_p = u$ for some particular $x_p \in X$, then describe all solutions for x in this case. What condition ensures that there will be *only* one solution?
   (e) What are the possibilities for the rank of A if it has a left inverse? What if it has a right inverse?