Read: Lax, Chapter 8, pages 101–120.

1. For any positive-definite self-adjoint mapping $M\colon X \to X$, define an inner product on X by $\langle x, y\rangle := (x, My)$. Throughout this problem, let $X = \mathbb{R}^2$ and $M = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$.
   (a) Find two orthonormal bases for X that contain the vector $e_1/||e_1||$, where $e_1 = (1, 0)$.
   (b) Find two orthonormal bases for X that contain the vector $e_2/||e_2||$, where $e_2 = (0, 1)$.
   (c) Find an vector v_2 orthogonal to $v_1 = (1, 1)$.
   (d) Find a matrix H that is self-adjoint with respect to $(\ ,\ )$, but *not* with respect to $\langle\ ,\ \rangle$.

2. Let $H, M\colon X \to X$ be self-adjoint mappings, and M positive-definite.
   (a) Formulate and prove a necessary and sufficient condition for $M^{-1}H$ to be self-adjoint with respect to the standard inner product.
   (b) Prove that $M^{-1}H$ is self-adjoint with respect to the inner product $\langle x, y\rangle = (x, My)$.
   (c) Prove that if H is positive-definite, then so is $M^{-1}H$.

3. Let $H, M\colon X \to X$ be self-adjoint mappings, and M positive definite. Define
$$R_{H,M}(x) = \frac{(x, Hx)}{(x, Mx)}\,.$$
   (a) Let $\mu = \min\{R_{H,M}(x) \mid x \in X\}$. Show that μ exists, and that the $v \in X$ for which $R_{H,M}(v) = \mu$ satisfies $Hv = \mu M v$.
   (b) Show that the constrained minimum problem
$$\min\big\{R_{H,M}(x) \mid (x, Mv) = 0\big\}$$
   has a nonzero solution $w \in X$, which satisfies $Hw = \kappa M w$, where $\kappa = R_{H,M}(w)$.

4. Let $H, M\colon X \to X$ be self-adjoint mappings, and M positive definite.
   (a) Show that there exists a basis $v_1, \dots, v_n$ of X where each v_i satisfies
$$Hv_i = \mu_i M v_i \quad (\mu_i \text{ real}), \qquad (v_i, Mv_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$
   (b) Compute (v_i, Hv_j), and show that there is an invertible matrix U for which $U^*MU = I$ and U^*HU is diagonal.
   (c) Characterize the numbers $\mu_1, \dots \mu_n$ by a minimax principle.