Linear maps

Matthew Macauley

Department of Mathematical Sciences
Clemson University
http://www.math.clemson.edu/~macaule/

Math 8530, Fall 2020

Preliminaries

Goal

Abstract the concept of a matrix as a linear mapping between vector spaces.

Advantages:

- simple, transparent proofs;
- better handles infinite dimensional spaces.

Definition

A linear map (or *mapping*, *transformation*, or *operator*) between vector spaces X and U over K is a function $T\colon X \to U$ that is:

(i) additive: $T(x+y) = T(x) + T(y)$, for all $x, y \in X$;

(ii) homogeneous: $T(ax) = aT(x)$, for all $x \in X$, $a \in K$.

The domain space is X and the target space is U.

Usually we'll write Tx for $T(x)$, and so the additive property is just the distributive law:

$$T(x+y) = Tx + Ty.$$

Examples of linear maps

Examples

(i) Any isomorphism;

(ii) $X = U = \{\text{polynomials of degree } < n \text{ in s}\}, \quad T = \frac{d}{ds}$.

(iii) $X = U = \mathbb{R}^2, \quad T =$ rotation about the origin.

(iv) X any vector space, $U = K$ (1-dimensional), T any $\ell \in X'$.

(v) $X = U = C_0(\mathbb{R}), \quad (Tf)(x) = \displaystyle\int_{-1}^{1} f(y)(x-y)^2\, dy.$

(vi) $X = \mathbb{R}^n$, $U = \mathbb{R}^m$, $u = Tx$, where $u_i = \displaystyle\sum_{j=1}^{n} t_{ij} x_j, \quad i = 1, \dots, m.$

(vii) $X = U = \big\{\text{piecewise cont. } [0,\infty) \to \mathbb{R} \text{ of ``exponential order''}\big\}$,
$(Tf)(s) = \displaystyle\int_0^\infty f(t) e^{-st}\, dt.$ "*Laplace transform*"

(viii) $X = U = \big\{\text{functions with } \int_{-\infty}^{\infty} |f(x)|\, dx < \infty\big\}$,
$(Tf)(\xi) = \displaystyle\int_{-\infty}^{\infty} f(x) e^{i\xi x}\, dx.$ "*Fourier transform*"

Basic properties

Theorem 3.1

Let $T\colon X \to U$ be a linear map.

(a) The image of a subspace of X is a subspace of U.

(b) The preimage of a subspace U is a subspace of X.

(Proof is a HW exercise.) □

Definition

The range of T is the image $R_T := T(X)$. The rank of T is $\dim R_T$.

The nullspace of T is the preimage of 0:

$$N_T := T^{-1}(0) = \{x \in X \colon Tx = 0\}.$$

The nullity of T is $\dim N_T$.

Rank-nullity theorem

Theorem 3.2

Let $T\colon X \to U$ be a linear map. Then $\dim N_T + \dim R_T = \dim X$.

Proof

Since T maps N_T to 0, then $Tx_1 = Tx_2$ if $x_1 \equiv x_2 \bmod N_T$.

Thus, T extends to a well-defined map on the quotient space X/N_T:

$$T\colon X/N_T \longrightarrow U\,, \qquad T\{x\} = Tx\,.$$

Note that this map is 1–1, and so $\dim(X/N_T) = \dim R_T$.

Therefore, $\dim X = \dim N_T + \dim X/N_T = \dim N_T + \dim R_T$. □

Consequences of the rank-nullity theorem

Corollary A

Suppose $\dim U < \dim X$. Then $Tx = 0$ for some $x \neq 0$.

Proof

We have $\dim R_T \leq \dim U < \dim X$, so by the R-N Theorem, $\dim N_T > 0$.

Thus, there is some nonzero $x \in N_T$. □

Example A

Take $X = \mathbb{R}^n$, $U = \mathbb{R}^m$, with $m < n$. Let $T \colon \mathbb{R}^n \to \mathbb{R}^m$ be any linear map (see Example (vi)).

Since $m = \dim U < \dim X < n$, Corollary A implies that the system of m equations

$$\sum_{j=1}^{n} t_{ij} x_j = 0 \qquad i = 1, \dots, m$$

has a non-trivial solution, i.e., not all $x_j = 0$.

Consequences of the rank-nullity theorem

Corollary B

Suppose $\dim X = \dim U$ and the only vector satisfying $Tx = 0$ is $x = 0$. Then $R_T = U$.

Proof

We have $N_T = \{0\}$, which means that $\dim N_T = 0$.

Clearly, $R_T \leq U$ ["*is a subspace of*"]. We just need to show they have the same dimension.

By the R-N Theorem, $\dim U = \dim X = \dim R_N + \dim N_T = \dim R_N$. $\square$

Example B

Take $X = U = \mathbb{R}^n$, and $T \colon \mathbb{R}^n \to \mathbb{R}^n$ given by $\sum_{j=1}^{n} t_{ij}x_j = u_i$, for $i = 1, \dots, n$.

If the related homogeneous system of equations $\sum_{j=1}^{n} t_{ij}x_j = 0$, for $i = 1, \dots, n$, has only the trivial solution $x_1 = \cdots x_n = 0$, then the inhomogeneous system T has a unique solution for all $u_1 \dots, x_n$.

[*Reason*: $T \colon \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism.]

Applications of the rank-nullity theorem

Application 1: Polynomial interpolation

Take $X = \{p \in \mathbb{C}[x] : \deg p < n\}$, $U = \mathbb{C}^n$, and let $s_1, \dots, s_n \in \mathbb{C}$ all be distinct. Define

$$T \colon X \to U\,, \qquad Tp = \big(p(s_1), \dots, p(s_n)\big)\,.$$

Suppose $Tp = 0$ for some $p \in X$. Then $p(s_1) = \cdots = p(s_n) = 0$, which is impossible because p has at most $n-1$ distinct roots.

Therefore $N_T = \{0\}$, and so Corollary B implies that $R_T = U$.

Application 2: Average values of polynomials

Let $X = \{p \in \mathbb{R}[x] : \deg p < n\}$, $U = \mathbb{R}^n$, and $I_1, \dots, I_n$ be pairwise disjoint intervals on $\mathbb{R}$.

The average value of p over I_j is the integral

$$\overline{p_j} := \frac{1}{|I_j|} \int_{I_j} p(s)\, ds\,.$$

Define $T \colon X \to U$ by $Tp = (\overline{p_1}, \dots, \overline{p_n})$.

Suppose $Tp = 0$. Then $\overline{p_j} = 0$ for all j, and so p (if nonzero) must change sign in I_j.

But this would imply that p has n distinct roots, which is impossible.

Thus, $N_T = \{0\}$, and so $R_T = U$.

Application to numerical analysis

Application 3: Numerical solutions to Laplace's equation

Laplace's equation is $\Delta u = u_{xx} + u_{yy} = 0$, where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is a linear operator.

Solutions to Laplace's PDE ("harmonic functions") are the functions in the nullspace of Δ.

If we fix the value of u on the boundary of a region $G \subset \mathbb{R}^2$, the solution to the boundary value problem $\Delta u = 0$ is as "flat as possible". [*Think*: plastic wrap stretched around ∂G.]

This models steady-state solutions to the heat equation PDE: $u_t = \Delta u$.

The finite difference method is a way to solve $\Delta u = 0$ numerically, using a square lattice with mesh spacing $h > 0$.

At a fixed lattice point O, let u_0 be the value of u at O, and u_W, u_E, u_N, u_S be the values at the neighbors.

We can approxmiate the derivatives with *centered differences*:

$$u_{xx} \approx \frac{u_W - 2u_0 + u_E}{h^2}, \qquad u_{yy} \approx \frac{u_N - 2u_0 + u_S}{h^2}.$$

Plugging this back into $\Delta u = 0$ gives $u_0 = \dfrac{u_W + u_N + u_E + u_S}{4}$, i.e., u_0 is the average of its four neighbors.

Application to numerical analysis (cont.)

Recall that we are trying to solve an inhomogeneous boundary value problem for Laplace's equation

$$\Delta u = 0 \,, \qquad u|_{\partial G} = f(x,y) \neq 0.$$

Claim

The homogeneous equation: $\Delta u = 0$, where $u = 0$ on ∂G, has *only* the trivial solution $u_0 = 0$ for all $(x,y) \in G$.

Proof

Let $\hat{O}$ be the lattice point at which u achieves its maximum value.

Since $u_0 = \dfrac{u_W + u_N + u_E + u_S}{4}$, then $u_0 = u_W = u_N = u_E = u_S$.

Repeating this, we see that *all* lattice points take the same value for u, and so $u = 0$.

By the result in Example B, the related inhomogenous system for $\Delta u = 0$, with arbitrary (non-zero) boundary conditions has a unique solution. □

Algebra of linear mappings

Definition

Let $S, T \colon X \to U$ be linear maps. Define

- $T + S$ by $(T + S)(x) = Tx + Sx$ for each $x \in X$.
- aT by $(aT)(x) = T(ax)$ for each $x \in X$, $a \in K$.

Easy fact

The set of linear maps from $X \to U$, denoted $\mathscr{L}(X, U)$, or $\mathrm{Hom}(X, U)$, is a vector space.

Theorem 3.3 (HW exercise)

If $T \colon X \to U$ and $S \colon U \to V$ are linear maps, then so is $(S \circ T) \colon X \to V$.

Moreover, composition is distributive w.r.t. addition. That is, if $P, T \colon X \to U$ and $R, S \colon U \to V$, then

$$(R + S) \circ T = R \circ T + S \circ T, \qquad S \circ (T + P) = S \circ T + S \circ P\,.$$

Remarks

- We usually just write $S \circ T$ as just ST.
- In general, $ST \neq TS$ (note that TS may not even be defined).

Invertibility

Definition

A linear map T is invertible if it is 1–1 and onto (i.e., if it is an isomorphism). Denote the inverse by T^{-1}.

Exercise

If T is invertible, then TT^{-1} is the identity.

Theorem 3.4 (exercise)

Let $T\colon X \to U$ be linear.

(i) If T is linear, then so is T^{-1}.

(ii) If S and T are invertible and ST defined, then it is invertible with $(ST)^{-1} = T^{-1}S^{-1}$.

Examples

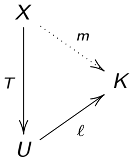
(ix) Take $X = U = V = \mathbb{R}[s]$, with $T = \frac{d}{ds}$ and $S =$ multiplication by s.

(x) Take $X = U = V = \mathbb{R}^3$, with S a 90°-rotation around the x_1 axis, and T a 90°-rotation around the x_2 axis.

In both of these examples, S and T are linear with $ST \neq TS$. (Which are invertible?)

Transposes

Let $T\colon X \to U$ be linear and $\ell \in U'$ (recall: $\ell\colon U \to K$).



The composition $m := \ell T$ is a linear map $X \to K$, i.e., an element of X'.

Since T is fixed, this defines an assignment of each $m \in X'$ to $\ell \in U'$.

This defines the following linear map, called the transpose of T:

$$T'\colon U' \longrightarrow X', \qquad T'\colon \ell \longmapsto m,$$

Using scalar product notation we can rewrite $m(x) = \ell(T(x))$ as $(m, x) = (\ell, Tx)$.

Key property

The transpose of $T\colon X \to U$ is the (unique) map $T'\colon U' \to X'$ that satisfies $m = T'\ell$, i.e.,

$$(T'\ell, x) = (\ell, Tx), \qquad \text{for all } x \in X,\ \ell \in U'.$$

Caveat: We are writing ℓT for $\ell \circ T$, but $T'\ell$ for $T'(\ell)$ (much like Tx for $T(x)$).

Properties (HW exercise)

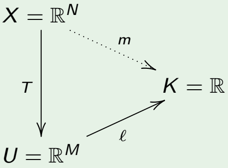
Whenever meaningful, we have

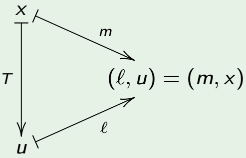
$$(ST)' = T'S', \qquad (T+R)' = T' + R', \qquad (T^{-1})' = (T')^{-1}.$$

Transposes

Examples (cont.)

(xi) Let $X = \mathbb{R}^N$, $U = \mathbb{R}^M$, and $Tx = u$, where $u_i = \sum_{j=1}^{N} t_{ij} x_j$.





By definition, for some $\ell_1, \ldots, \ell_m \in K$,

$$(\ell, u) = \sum_{i=1}^{M} \ell_i u_i = \sum_{i=1}^{M} \ell_i \left(\sum_{j=1}^{N} t_{ij} x_j \right) = \sum_{i=1}^{M} \sum_{j=1}^{N} \ell_i t_{ij} x_j = \sum_{i=1}^{N} \left(\ell_i \sum_{j=1}^{M} t_{ij} x_j \right) = \sum_{j=1}^{N} m_j x_j$$

This gives us a formula for $m = (m_1, \ldots, m_N)$, where $(\ell, u) = (m, x)$.

We'll see later that if we express T in matrix form, then T' is formed by making the rows of T the columns of T'.

Transposes

Proposition

If X'' and U'' are canonically identified with X and U, respectively, then $T'' = T$. □

Theorem 3.5

The annihilator of the range of T is the nullspace of its transpose, i.e., $R_T^\perp = N_{T'}$.

Proof

By definition,

$$\begin{aligned} R_T^\perp &= \{\ell \in U' : (\ell, u) = 0 \ \ \forall u \in R_T\} \\ &= \{\ell \in U' : (\ell, Tx) = 0 \ \ \forall x \in X\} \\ &= \{\ell \in U' : (T'\ell, x) = 0 \ \ \forall x \in X\} \\ &= N_{T'}\,. \end{aligned}$$

Thus, $\ell \in R_T^\perp$ iff $T'\ell = 0$, i.e., iff $\ell \in N_{T'}$. □

Applying $\perp$ to both sides of $R_T^\perp = N_{T'}$ (Theorem 3.5) yields the following:

Corollary 3.5

The range of T is the annihilator of the nullspace of T', i.e., $R_T = N_{T'}^\perp$. □

Transposes

Theorem 3.6

For any linear mapping $T\colon X \to U$, we have $\dim R_T = \dim R_{T'}$.

Proof

We can deduce the following easy facts:

- $\dim R_T^\perp + \dim R_T = \dim U$ (Theorem 2.4 applied to $R_T \subseteq U$);
- $\dim N_{T'} + \dim R_{T'} = \dim U'$ (R-N Theorem applied to $T'\colon U' \to X'$);
- $\dim U = \dim U'$ (Theorem 2.2).

Now, $R_T^\perp = N_{T'}$ (Theorem 3.5) immediately yields the result. □

Corollary 3.6

Let $T\colon X \to U$ be linear with $\dim X = \dim U$. Then $\dim N_T = \dim N_{T'}$.

Proof

Apply the R-N Theorem to $T\colon X \to U$ and $T'\colon U' \to X'$:

- $\dim N_T = \dim X - \dim R_T$;
- $\dim N_{T'} = \dim U' - \dim R_{T'}$.

Now apply $\dim X = \dim U = \dim U'$ (assumption), and $\dim R_T = \dim R_{T'}$ (Theorem 3.6). □

Algebra of linear mappings, revisited

Definition

An endomorphism of a vector space X is a linear map from X to itself. Denote the set of endomorphisms of X by $\mathscr{L}(X, X)$ or $\text{Hom}(X, X)$ or $\text{End}(X)$.

Remarks

$\mathscr{L}(X, X)$ is a vector space, but we can also "multiply" vectors; it is an algebra.

It is an associative but noncommutative algebra, with unity I, satisfying $Ix = x$.

$\mathscr{L}(X, X)$ contains zero divisors: pairs S, T such that $ST = 0$ buth neither S nor T is zero.

Proposition

If $A \in \mathscr{L}(X, X)$ is a left inverse of $B \in \mathscr{L}(X, X)$ [i.e., $AB = I$], then it is also a right inverse [i.e., $BA = I$]. □

Definition

The invertible elements of $\mathscr{L}(X, X)$ forms the general linear group, denoted $\text{GL}(n, K)$, where $n = \dim X$.

Every $S \in \text{GL}(n, K)$ defines a similarity transformation of $\mathscr{L}(X, X)$, sending $M \longmapsto M_S := SMS^{-1}$, for each $M \in \mathscr{L}(X, X)$. We say M and M_S are similar.

Similarity

Theorem 3.7

Every similarity transform is an automorphism ["structure-preserving bijection"] of $\mathscr{L}(X, X)$:

$$(kM)_S = kM_S, \qquad (M+N)_S = M_S + N_S, \qquad (MN)_S = M_S N_S\,.$$

Moreover, the set of similarity transforms forms a group under $(M_S)_T := M_{TS}$, called the inner automorphism group of $GL(n, K)$.

Proof

Verification of $(kM)_S = kM_S$, and $(M+N)_S = M_S + N_S$ is trivial.

Next, observe that $M_S N_S = (SMS^{-1})(SNS^{-1}) = SMNS^{-1} = (MN)_S$.

Finally, $(M_S)_T = T(SMS^{-1})T^{-1} = (TS)M(TS)^{-1} = M_{TS}$.

Checking the group axioms is a straight-forward exercise. □

Theorem 3.8 (exercise)

Similarity is an equivalence relation, i.e., it is:

(i) Reflexive: $M \sim M$;

(ii) Symmetric: $L \sim M$ implies $M \sim L$;

(iii) Transitive: $L \sim M$ and $M \sim N$ implies $L \sim N$. □

Algebra of linear mappings

Theorem 3.9 (HW exercise)

If either A or B in $\mathscr{L}(X,X)$ is invertible, then AB and BA are similar. □

Given any $A \in \mathscr{L}(X,X)$ and polynomial $p(s) = a_N s^N + \cdots + a_1 s + a_0$, consider the polynomial $p(A) = a_N A^N + \cdots + a_1 A + a_0 I$.

The set of polynomials in A is a commutative subalgebra of $\mathscr{L}(X,X)$. [to be revisited]

Miscellaneous definitions

- A linear map $P\colon X \to X$ is a projection if $P^2 = P$.
- The commutator of $A, B \in \mathscr{L}(X,X)$ is $[A,B] := AB - BA$, which is 0 iff A and B commute.

Examples (cont.)

(xii) If $X = \{f : \mathbb{R} \to \mathbb{R},\ \text{contin.}\}$, then the following maps $P, Q \in \mathscr{L}(X,X)$ are projections:

- $(Pf)(x) = \dfrac{f(x) + f(-x)}{2}$; this is the even part of f.
- $(Qf)(x) = \dfrac{f(x) - f(-x)}{2}$; this is the odd part of f.

Note that $f = Pf + Qf$ for any $f \in X$.