Lecture 2.2: Applications of the rank-nullity theorem

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Overview

In the last lecture, we learned about a fundamental result of linear maps.

Rank-nullity theorem

Let $T\colon X \to U$ be a linear map. Then $\dim R_T + \dim N_T = \dim X$.

In this lecture, we will show how this theoretical result has surprising implications, involving polynomials, ODEs, and PDEs.

We will also use the following simple corollary from the previous lecture:

Corollary B

Suppose $\dim X = \dim U < \infty$ and the only vector satisfying $Tx = 0$ is $x = 0$. Then $R_T = U$.

Polynomial interpolation

Let $X = \{p \in \mathbb{C}[x] \mid \deg p < n\}$ and $U = \mathbb{C}^n$.

Pick any distinct $s_1, \ldots, s_n \in \mathbb{C}$, and define

$$T \colon X \longrightarrow U, \qquad T \colon p \mapsto \big(p(s_1), \ldots, p(s_n)\big).$$

Suppose $Tp = 0$ for some $p \in X$.

Then $p(s_1) = \cdots = p(s_n) = 0$, which is impossible because p has at most $n-1$ distinct roots.

Therefore $N_T = \{0\}$, and so Corollary B implies that $R_T = U$.

Average value of a polynomial

Let $X = \{p \in \mathbb{R}[x] \mid \deg p < n\}$ and $U = \mathbb{R}^n$.

Let $I_1, \dots, I_n \subseteq \mathbb{R}$ be pairwise disjoint intervals.

The average value of p over I_j is

$$\overline{p_j} := \frac{1}{|I_j|} \int_{I_j} p(t)\, dt.$$

Define the linear function

$$T \colon X \longrightarrow U, \qquad Tp = (\overline{p_1}, \dots, \overline{p_n}).$$

Suppose $Tp = 0$. Then $\overline{p_j} = 0$ for all j, and so any nonzero p must change sign in I_j.

But this would imply that p has n distinct roots, which is impossible.

Thus, $N_T = \{0\}$, and so $R_T = U$.

Systems of equations

Our next two applications will rely on the following result from the previous lecture.

Example B

Take $X = U = \mathbb{R}^n$, and $T\colon \mathbb{R}^n \to \mathbb{R}^n$ given by $\displaystyle\sum_{j=1}^{n} t_{ij}x_j = u_i$, for $i = 1, \dots, n$.

If the related homogeneous system of equations $\displaystyle\sum_{j=1}^{n} t_{ij}x_j = 0$, for $i = 1, \dots, n$, has only the trivial solution $x_1 = \cdots x_n = 0$, then the inhomogeneous system T has a unique solution.

Recall that this followed from:

Corollary B

Suppose $\dim X = \dim U$ and the only vector satisfying $Tx = 0$ is $x = 0$. Then $R_T = U$.

ODEs: the method of undetermined coefficients

Consider the differential equation

$$\underbrace{ay'' + by' + cy}_{\text{homogeneous part}} = \underbrace{5e^{3t}\cos 4t}_{\text{"forcing term", } f(t)}$$

In an ODEs class, you learn that the general solution has the form $y(t) = y_h(t) + y_p(t)$.

Here, $y_h(t)$ is the general solution to the homogeneous equation $ay'' + by' + cy = 0$, i.e., the nullspace of

$$L \colon \mathcal{C}^\infty(\mathbb{R}) \longrightarrow \mathcal{C}^\infty(\mathbb{R}), \qquad L \colon y \longmapsto ay'' + by' + cy.$$

If the forcing term $f(t) = 5e^{3t}\cos 4t$ doesn't solve the homogeneous equation, we can find a "particular solution" of the form $y_p(t) = Ae^{3t}\cos 4t + Be^{3t}\sin 4t$.

But *why* does this work? Let $X = \text{Span}(e^{3t}\cos 4t,\ e^{3t}\sin 4t)$.

The only solution to the homogeneous equation $Ly = 0$ in X is $y = 0$.

We are trying to solve the inhomogeneous equation $Ly = f$, and $f \in X$.

By Example B, there is a unique $y_p \in X$ satisfying $Ly_p = f$.

PDEs: numerical solutions to Laplace's equation

Laplace's equation is $\Delta u = u_{xx} + u_{yy} = 0$, where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is a linear operator.

Solutions to Laplace's PDE ("harmonic functions") are the functions in the nullspace of Δ.

If we fix the value of u on the boundary of a region $G \subset \mathbb{R}^2$, the solution to the boundary value problem $\Delta u = 0$ is as "flat as possible". [*Think*: plastic wrap stretched around ∂G.]

This models steady-state solutions to the heat equation PDE: $u_t = \Delta u$.

The finite difference method is a way to solve $\Delta u = 0$ numerically, using a square lattice with mesh spacing $h > 0$.

At a fixed lattice point O, let u_0 be the value of u at O, and u_W, u_E, u_N, u_S be the values at the neighbors.

We can approximate the derivatives with *centered differences*:

$$u_{xx} \approx \frac{u_W - 2u_0 + u_E}{h^2}, \qquad u_{yy} \approx \frac{u_N - 2u_0 + u_S}{h^2}.$$

Plugging this back into $\Delta u = 0$ gives $u_0 = \dfrac{u_W + u_N + u_E + u_S}{4}$, i.e., u_0 is the average of its four neighbors.

Numerical solutions to Laplace's equation (contin.)

Recall that we are trying to solve an inhomogeneous boundary value problem for Laplace's equation

$$\Delta u = 0\,, \qquad u|_{\partial G} = f(x,y) \neq 0.$$

Claim

The homogeneous equation: $\Delta u = 0$, where $u = 0$ on ∂G, has *only* the trivial solution $u_0 = 0$ for all $(x,y) \in G$.

Proof (sketch)

Let $\hat{O}$ be the lattice point at which u achieves its maximum value.

Since $u_0 = \dfrac{u_W + u_N + u_E + u_S}{4}$, then $u_0 = u_W = u_N = u_E = u_S$.

Repeating this, we see that *all* lattice points take the same value for u, and so $u = 0$.

By the result in Example B, the related inhomogeneous system for $\Delta u = 0$, with arbitrary (non-zero) boundary conditions has a unique solution. □