Lecture 2.6: Matrices

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How to encode a linear map with a matrix

Let $T \colon X \to U$ be a linear map between finite-dimensional vector spaces.

To encode T as a matrix, we'll need to choose:

1. an "input basis" $\mathcal{B}_X = \{x_1, \dots, x_n\}$ for X,
2. an "output basis" $\mathcal{B}_U = \{u_1, \dots, u_m\}$ for U.

Let $\{\ell_1, \dots, \ell_m\}$ be the dual basis of $\mathcal{B}_U$.

First, we write the images of the basis vectors in $\mathcal{B}_X$ using the basis vectors in $\mathcal{B}_U$:

$$
\begin{aligned}
Tx_1 &= \\
Tx_2 &= \\
&\vdots \\
Tx_j &= \\
&\vdots \\
Tx_n &=
\end{aligned}
$$

Summary

Let $T\colon X \to U$. The matrix A of T w.r.t. bases $\mathcal{B}_X = \{x_1, \dots, x_n\}$ and $\mathcal{B}_U = \{u_1, \dots, u_m\}$ is

$$A = {}_{\mathcal{B}_X}\big[T\big]_{\mathcal{B}_U} = \Big[Tx_1 \;\; Tx_2 \;\; \cdots \;\; Tx_n\Big].$$

Remarks

- The range of T is the span of the column vectors – the column space.
- $a_{ij} = (\ell_i, Tx_j)$,

$$\begin{aligned}
Tx_1 &= a_{11}u_1 + a_{21}u_1 + \cdots + a_{i1}u_j + \cdots + a_{m1}u_m \\
Tx_2 &= a_{12}u_1 + a_{22}u_1 + \cdots + a_{i2}u_j + \cdots + a_{m2}u_m \\
&\;\;\vdots \\
Tx_j &= a_{1j}u_1 + a_{2j}u_1 + \cdots + a_{ij}u_j + \cdots + a_{mj}u_m \\
&\;\;\vdots \\
Tx_n &= a_{1n}u_1 + a_{2n}u_1 + \cdots + a_{in}u_j + \cdots + a_{mn}u_m
\end{aligned}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Example 1

Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the projection onto the line $y = x$.

An interesting choice of basis

Proposition

If $T\colon X \to U$ is invertible, we can always choose $\mathcal{B}_X$ and $\mathcal{B}_U$ so the matrix is the identity.

More generally, for any $T\colon X \to U$, we can choose $\mathcal{B}_X$ and $\mathcal{B}_U$ so the matrix in block form is

$$A = {}_{\mathcal{B}_X}\big[T\big]_{\mathcal{B}_U} = \begin{bmatrix} I_{r\times r} & 0 \\ 0 & 0 \end{bmatrix}.$$

Example 2

Let $X = \{c_0 + c_1 x + c_2 x^2 \mid c_i \in \mathbb{R}\}$ with basis $\mathcal{B}_X = \{1, x, x^2\}$.

Let $U = \{c_0 + c_1 x \mid c_i \in \mathbb{R}\}$ with basis $\mathcal{B}_U = \{1, x\}$.

Let $T = \dfrac{d}{dx}$, and so $T \colon c_0 + c_1 x + c_2 x^2 \mapsto c_1 + 2c_2 x$.

The matrix of the transpose

Let $T\colon X \to U$ be linear, and pick bases $\mathcal{B}_X = \{x_1, \dots, x_n\}$ and $\mathcal{B}_U = \{u_1, \dots, u_m\}$.

Let $\mathcal{B}_{U'} = \{\ell_1, \dots, \ell_m\}$ be the dual basis of $\mathcal{B}_U$.

Let $A = (a_{ij})$ be the matrix of T w.r.t. these bases.

In plain English, a_{ij} is the result of:

1. starting with the $j^{\rm th}$ basis vector in X,
2. applying the map T,
3. applying the $i^{\rm th}$ dual basis vector in U'.

Let's apply these steps to the transpose map $T'\colon U' \to X'$ to find its matrix form, $A' = (a'_{ij})$.