Lecture 3.5: The determinant and trace of a matrix

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The determinant of a 2×2 matrix

The determinant of an $n \times n$ matrix can be thought of as an alternating n-linear function of its column vectors.

Let's use bilinearity to find a formula for the determinant of $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$.

The determinant of a 3×3 matrix

Let's now apply this to finding the determinant of $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

The determinant of an $n \times n$ matrix

Proposition 3.8

The determinant of an $n \times n$ matrix $A = (a_{ij})$ is

$$\det A = \sum_{\pi \in S_n} a_{1,\pi(1)} a_{2,\pi(2)} \cdots a_{n,\pi(n)},$$

and by symmetry, $\det A = \det A^T$.

The trace of a matrix

Definition

The trace of an $n \times n$ matrix is $\operatorname{tr} A = \sum_{i=1}^{n} a_{ii}$.

Proposition 3.9

(a) Trace is linear: $\operatorname{tr}(kA) = k(\operatorname{tr} A)$ and $\operatorname{tr}(A + B) = \operatorname{tr} A + \operatorname{tr} B$.

(b) Trace is "commutative": $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

(c) Similar matrices have the same determinant and trace.