Lecture 4.2: The Cayley-Hamilton theorem

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Definitions

Throughout, $A\colon X \to X$ will be an $n \times n$ matrix over an algebraically closed field K.

Definition

The characteristic polynomial of A is

$$p_A(t) = \det(tI - A).$$

$$\det(tI - A) = \begin{vmatrix} t - a_{11} & -a_{12} & -a_{13} & \cdots & -a_{1(n-1)} & -a_{1n} \\ -a_{21} & t - a_{22} & -a_{23} & \cdots & -a_{2(n-1)} & -a_{2n} \\ -a_{31} & -a_{32} & t - a_{33} & \cdots & -a_{3(n-1)} & -a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_{(n-1)1} & -a_{(n-1)2} & -a_{(n-1)3} & \cdots & t - a_{(n-1)(n-1)} & -a_{(n-1)n} \\ -a_{n1} & -a_{n2} & -a_{n3} & \cdots & -a_{n(n-1)} & t - a_{nn} \end{vmatrix}$$

Remarks

- Recall that $\det M = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) m_{\pi(1),1} m_{\pi(2),2} \cdots m_{\pi(n),n}$.
- The characteristic polynomial has degree n, and its roots are the eigenvalues of A.

Determinant and trace, revisited

Proposition 4.4

If the eigenvalues of A are $\lambda_1, \dots, \lambda_n$, then

$$\operatorname{tr} A = \sum_{i=1}^{n} \lambda_i, \qquad \det A = \prod_{i=1}^{n} \lambda_i$$

This follows from the following two observations:

$$\det(tI - A) = \begin{vmatrix} t - a_{11} & -a_{12} & -a_{13} & \cdots & -a_{1(n-1)} & -a_{1n} \\ -a_{21} & t - a_{22} & -a_{23} & \cdots & -a_{2(n-1)} & -a_{2n} \\ -a_{31} & -a_{32} & t - a_{33} & \cdots & -a_{3(n-1)} & -a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_{(n-1)1} & -a_{(n-1)2} & -a_{(n-1)3} & \cdots & t - a_{(n-1)(n-1)} & -a_{(n-1)n} \\ -a_{n1} & -a_{n2} & -a_{n3} & \cdots & -a_{n(n-1)} & t - a_{nn} \end{vmatrix}$$

$$\det M = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) m_{\pi(1),1} m_{\pi(2),2} \cdots m_{\pi(n),n}.$$

Polynomials of matrices

Remark

If $Av = \lambda v$, then $A^k v = \lambda^k v$ for all $k \in \mathbb{N}$.

Actually, much more is true:

Spectral mapping theorem

If λ is an eigenvalue of A, then for any polynomial $q(t)$,

(a) $q(\lambda)$ is an eigenvalue of $q(A)$

(b) conversely, every eigenvalue of $q(A)$ has this form.

Corollary 4.5

Every eigenvalue of $p_A(A)$ is zero.

Actually, much more is true:

Cayley-Hamilton theorem

Every matrix satisfies its characteristic polynomial. That is, $p_A(A) = 0$.

Lemma 4.6 (exercise)

Let P and Q be polynomials with matrix coefficients:

$$P(t) = P_n t^n + \cdots + P_1 t + P_0, \qquad Q(t) = Q_m t^m + \cdots + Q_1 t + Q_0.$$

Their product is a polynomial

$$\begin{aligned} R(t) = P(t)Q(t) &= (P_n t^n + \cdots + P_1 t + P_0)(Q_m t^m + \cdots + Q_1 t + Q_0) \\ &= R_{n+m} t^{n+m} + \cdots + R_1 t + R_0, \end{aligned}$$

where $R_k = \sum_{i+j=k} P_i Q_j$. Moreover, if A commutes with the Q_i's, then $P(A)Q(A) = R(A)$.

We will apply this to the polynomial $Q(t) = tI - A$, and so $\det Q(t) = p_A(t)$.

Let C_{ji} be the (j,i) cofactor of $Q(t)$. By Cramer's theorem, $\det Q(t)I = (C_{ji})Q(t)$.

If we let $P(t) = (C_{ji})$, then

$$R(t) := P(t)Q(t) = \det Q(t)I = p_A(t)I.$$

Clearly, A commutes with the coefficients of $Q(t)$, and $Q(A) = 0$, so

$$R(A) = P(A)Q(A) = \det Q(A)I = p_A(A) = 0.$$