Lecture 4.5: The spectral theorem

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Overview and motivation

Throughout, assume K is algebraically closed, and $\dim X = n$. A generalized eigenvector of A is any $v \in X$ such that $(A-\lambda I)^m v = 0$ for some $m \geq 1$.

Spectral theorem

Let $A \colon X \to X$ be linear. Then X has a basis of generalized eigenvectors of A.

Recall our running example, a linear map with $p_A(t) = (t-\lambda)^{11}$, and $\dim N_{A-\lambda I} = 4$:

$$v_5 \xmapsto{A-\lambda I} v_4 \xmapsto{A-\lambda I} v_3 \xmapsto{A-\lambda I} v_2 \xmapsto{A-\lambda I} v_1 \xmapsto{A-\lambda I} 0$$

$$w_3 \xmapsto{A-\lambda I} w_2 \xmapsto{A-\lambda I} w_1 \xmapsto{A-\lambda I} 0$$

$$x_2 \xmapsto{A-\lambda I} x_1 \xmapsto{A-\lambda I} 0$$

$$y_1 \xmapsto{A-\lambda I} 0$$

If $N_j := N_{(A-\lambda I)^j}$, then

$$\cdots = N_6 = N_5 \supsetneq N_4 \supsetneq N_3 \supsetneq N_2 \supsetneq N_1 \supsetneq 0.$$

Supporting lemmas

Lemma 4.8

Let $p, q \in K[t]$ be co-prime. Then we can write $ap + bq = 1$ for some $a, b \in K[t]$.

Lemma 4.9

Let $A \colon X \to X$, and $p, q \in K[t]$ be co-prime. If N_p, N_q, N_{pq} are the nullspaces of $p(A)$, $q(A)$, and $p(A)q(A)$, then

$$N_{pq} = N_p \oplus N_q.$$

Corollary 4.10

If $p_1, \dots, p_k \in K[t]$ are pairwise co-prime, and $N_{p_1 \cdots p_k}$ is the nullspace of $p_1(A) \cdots p_k(A)$, then

$$N_{p_1 \cdots p_k} = N_{p_1} \oplus \cdots \oplus N_{p_k}.$$

Generalized eigenspaces

Definition

Let λ be an eigenvalue of A, an $n \times n$ matrix over K, with index $d_\lambda = \text{index}(\lambda)$. The generalized eigenspace of λ is

$$E_\lambda := N_{(A-\lambda I)^{d_\lambda}} = \bigcup_{j=1}^{\infty} N_{(A-\lambda I)^j}.$$

Spectral theorem (stronger)

Let $A\colon X \to X$ be linear, with distinct eigenvalues $\lambda_1, \dots, \lambda_k$. Then

$$X = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k}.$$