Lecture 5.3: Gram-Schmidt and orthogonal projection

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Constructing an orthonormal basis

Throughout, assume that X is an n-dimensional inner product space.

In the last lecture, we showed why having an orthogonal (or even better: orthonormal) basis is very convenient.

We'll start this lecture by showing how to *construct* an orthogonal basis.

Gram-Schmidt process

Given an arbitrary basis $x_1, \dots, x_n$, construct an orthonormal basis $q_1, \dots, q_n$ for which $q_k \in \text{Span}(x_1, \dots, x_k)$.

Remark

In matrix form, this leads to the QR factorization:

$$A = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix} \begin{bmatrix} \langle x_1, q_1 \rangle & \langle x_2, q_1 \rangle & \langle x_3, q_1 \rangle & \cdots \\ 0 & \langle x_2, q_2 \rangle & \langle x_3, q_2 \rangle & \cdots \\ 0 & 0 & \langle x_3, q_3 \rangle & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = QR.$$

Identifying a space with its dual

Earlier in this class, we found it helpful to think of dual vectors $\ell \in X'$ as row vectors.

Going forward, it will be helpful to canonically identify these elements with vectors in X.

However, *the isomorphism will depend on the inner product*.

Proposition 5.2

Every linear function $\ell \in X'$ can be written as

$$\ell(x) = \langle x, y \rangle, \qquad \text{for some fixed } y \in X.$$

Corollary 5.3

For any fixed $y \in X$, the mapping

$$R_y \colon X \longrightarrow X', \qquad R_y \colon y \longmapsto \langle -, y \rangle$$

is an isomorphism. There is an analogous isomorphism

$$L_x \colon X \longrightarrow X', \qquad L_x \colon x \longmapsto \langle x, - \rangle.$$

Orthogonal complements

Definition

Let Y be a subspace of X. The orthogonal complement of Y is the set

$$Y^{\perp} := \big\{x \in X \mid \langle x, y\rangle = 0, \ \ \forall y \in Y\big\}.$$

Proposition 5.4

For any subspace Y of X, we have $X = Y \oplus Y^{\perp}$.

Examples of orthogonal complements

Let's return to several familiar examples.

1. $X = \mathbb{R}^n$, with the standard dot product.

2. $X = \mathbb{R}^2$, with inner product

$$\big\langle a_1 e_1 + a_2 e_2, b_1 e_1 + b_2 e_2 \big\rangle = \begin{bmatrix} b_1 & b_2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 2a_1 b_1 + a_1 b_2 + b_1 a_2 + 2a_2 b_2.$$

3. $V = \mathrm{Hom}(X, Y)$ with inner product

$$\langle A, B \rangle = \mathrm{tr}(B^T A) = \sum_{i,j} a_{ij} b_{ij}.$$

4. $X = \mathrm{Per}_{2\pi}(\mathbb{R})$, the 2π-periodic functions, with the inner product

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) g(x)\, dx.$$

Orthogonal projection

If $X = Y \oplus Y^{\perp}$, then the map

$$P_Y \colon X \longrightarrow X, \qquad P_Y \colon y + y^{\perp} \longmapsto y$$

is the orthogonal projection of X onto Y.

Proposition 5.5 (exercise)

The orthogonal projection map P_Y is linear and idempotent (i.e., $P_Y^2 = P_Y$), and hence diagonalizable.

Proposition 5.6

The orthogonal projection map $P_Y \colon X \longrightarrow X$ sends $x \in X$ to

$$P_Y(x) = \arg\min \left\{ ||x - y|| : y \in Y \right\}.$$