

## Lecture 5.5: Isometries

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## Constructing an orthonormal basis

Roughly speaking, an isometry is a distance-preserving map.

### Definition

Let  $X$  be an inner product space. A function  $A: X \rightarrow X$  is an **isometry** if

$$\|Ax - Ay\| = \|x - y\|, \quad \text{for all } x, y \in X.$$

### Examples

The following are all isometries of  $\mathbb{R}^n$ :

1. any **translation**
2. any **rotation**
3. any **reflection**
4. any compositions of these.

The isometries of  $X$  form a group ... but that's not a group we're all that interested in.

## Orthogonal maps

Given any isometry, one can compose it with a translation to get an isometry that fixes 0.

Conversely, *any* isometry can be decomposed into one that fixes 0, followed by a translation.

### Definition

An isometry  $A: X \rightarrow X$  fixing 0 is said to be **orthogonal**.

The orthogonal maps on  $X$  form a group called the **orthogonal group**, denoted  $O(X)$ .

If  $X = \mathbb{R}^n$ , we denote this by  $O(n)$  or  $O_n$ .

We will say that a matrix **orthogonal** if it represents an orthogonal linear map.

### Remark

A matrix  $A$  is **orthogonal** if and only if its columns are **orthonormal**. That is, if  $A^T A = I$ .

Next, we'll show that all orthogonal maps are linear.

# Properties of orthogonal maps

## Theorem 5.11

Let  $A: X \rightarrow X$  be orthogonal.

- (i)  $A$  is linear
- (ii)  $A^*A = I$  (and conversely)
- (iii)  $A$  is invertible, and  $A^{-1}$  is an isometry
- (iv)  $\det A = \pm 1$ .

## Key point

The geometric meaning of this theorem is that any map fixing 0 that preserves **distances** is linear, preserves angles, and preserves volume.

## Definition

The subgroup of  $SO(X)$  of maps with determinant 1 is the **special orthogonal group**, denoted  $SO(X)$ .

Elements in  $SO(X)$  describe **rotations**.