Lecture 5.6: The norm of a linear map

Matthew Macauley

School of Mathematical & Statistical Sciences
Clemson University
http://www.math.clemson.edu/~macaule/

Math 8530, Advanced Linear Algebra

Overview

The norm of a vector measures its size, or magnitude.

The set $\text{Hom}(X, U)$ of linear maps is a vector space. So what is the norm of $A\colon X \to U$?

The determinant is one way to measure the "size" of a linear map. However, this won't work, because

1. it is only defined when $X = U$,
2. it cannot be a norm, as there are nonzero linear maps with determinant zero.

There are a number of approaches that will work. Two reasonable ones are

1. the norm arising from the inner product $\langle A, B\rangle := \text{tr}(B^* A)$,
2. the largest factor that A can stretch a vector.

Let's recall the following definition from real analysis.

Definition

The supremum of a bounded subset $S \subseteq \mathbb{R}$, is its least upper bound. This always exists, and is denoted $\sup S$.

Moreover, if S is closed (contains all of its limit points), then $\sup S = \max S$.

Frobenius and induced norms

We can define an inner product on $\text{Hom}(X, U)$ by

$$\langle A, B \rangle = \text{tr}(B^* A).$$

Naturally, this gives us a definition of the norm of a linear map.

Definition

Let X and U be vector spaces. The Frobenius norm of $A \colon X \to U$ is

$$||A|| = \sqrt{\text{tr}(A^* A)} = \sqrt{\sum_{i,j} |a_{ij}|^2}.$$

This does *not* depend on any inner product structure of X or U.

Alternatively, we can define $||A||$ as the largest factor that A stretches a (nonzero) vector by.

Clearly, this depends on the inner products (and hence norms) on X and U.

Definition

Let X and U be inner product spaces. The induced norm of $A \colon X \to U$ is

$$||A|| := \sup_{||x||=1} ||Ax|| = \sup_{x \neq 0} \frac{||Ax||}{||x||}.$$

Properties of the induced norm

Henceforth, we will use the induced norm, unless otherwise stated.

Proposition 5.12

For any linear map $A\colon X \to U$,

(i) $||Az|| \leq ||A|| \cdot ||z||$, for all $z \in X$.

(ii) $||A|| = \sup\limits_{||x||=||v||=1} \langle Ax, v \rangle$.

Properties of the induced norm

Proposition 5.13

Given linear maps $A, B\colon X \to U$ and $C\colon U \to V$,

(i) $||kA|| = |k| \cdot ||A||$

(ii) $||A + B|| \leq ||A|| + ||B||$

(iii) $||CA|| \leq ||C|| \cdot ||A||$

(iv) $||A^*|| = ||A||$.

Open sets and invertible maps

Let X be a vector space with a norm. For $x_0 \in X$ and $r > 0$, define the ball of radius r, centered at x to be

$$B_r(x_0) = \{x \in X : ||x - x_0|| < r\}.$$

A subset $U \subseteq X$ is open if for every $u \in U$, there is some $r > 0$ for which $B_r(u) \subseteq U$.

The following implies that the subset of invertible maps is open.

Theorem 5.14

Let $A \colon X \to U$ be invertible, and suppose $B \colon X \to U$

$$||A - B|| < \frac{1}{||A^{-1}||}.$$

Then B is invertible.

Other norms

Definition

Let X and U be a vector spaces over R. A norm on $\text{Hom}(X, U)$ is a function

$$||\cdot|| : \text{Hom}(X, U) \longrightarrow \mathbb{R}$$

such that

1. $||kA|| = |k| \cdot ||A||$
2. $||A + B|| \leq ||A|| + ||B||$
3. $||A|| > 0$ for $A \neq 0$.

If $X = U$, then a norm is submultiplicative if

$$||AB|| \leq ||A|| \cdot ||B||.$$