# Lecture 5.8: Complex inner product spaces

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# Real vs. complex vector spaces

We have primarily been dealing with $\mathbb{R}$-vector spaces. Things are a little different over $\mathbb{C}$.

Let's compare the notion of *norm* for real vs. complex numbers.

- For any real number $x \in \mathbb{R}$, its norm (distance from 0) is $|x| = \sqrt{x^2} \in \mathbb{R}$.
- For any complex number $z = a + bi \in \mathbb{C}$, its norm (distance from 0) is defined by

$$|z| = \sqrt{z\overline{z}} = \sqrt{(a+bi)(a-bi)} = \sqrt{a^2+b^2}.$$

Let's now go from $\mathbb{R}$ and $\mathbb{C}$ to $\mathbb{R}^2$ and $\mathbb{C}^2$.

- For any vector $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$, its norm (distance from 0) is

$$||x|| = \sqrt{\langle x, x \rangle} = \sqrt{x^T x} = \sqrt{x_1^2 + x_2^2}.$$

- For any $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in \mathbb{C}^2$, with $z_1 = a + bi$, $z_2 = c + di$, its norm is defined by

$$||z|| = \sqrt{\langle z, z \rangle} := \sqrt{\overline{z}^T z} = \sqrt{|z_1|^2 + |z_2|^2}.$$

For example, let's compute the norms of $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in \mathbb{R}^2$ and $z = \begin{bmatrix} i \\ i \end{bmatrix} \in \mathbb{C}^2$.

# Complex dot product

## Definition

If $X$ is a finite-dimensional vector space of $\mathbb{C}$, then define the complex dot product as

$$\langle z, w \rangle = \overline{w}^H z := \overline{w}^T z = \begin{bmatrix} \overline{w_1} & \overline{w_2} & \cdots & \overline{w_n} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}.$$

Here, $H$ stands for Hermitian.

The norm of a vector $z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$ in $\mathbb{C}^n$ is thus defined by

$$||z||^2 = \langle z, z \rangle = \overline{z}^T z = \begin{bmatrix} \overline{z_1} & \overline{z_2} & \cdots & \overline{z_n} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2.$$

Just like how we abstracted the dot product to a real inner product, we can abstract the complex dot product to a complex inner product.

# Complex inner products and sesquilinear forms

## Definition

A complex inner product space is a vector space $X$ over $\mathbb{C}$ endowed with a map

$$\langle\ ,\ \rangle : X \times X \longrightarrow \mathbb{C}$$

satisfying

(i) $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ and $\langle u, v+w\rangle = \langle u, v\rangle + \langle u, w\rangle$

(ii) $\langle ku, v\rangle = k\langle u, v\rangle$ "*linear in the 1st coordinate*"

(iii) $\langle u, kv\rangle = \overline{k}\langle u, v\rangle$ "*antilinear in the 2nd coordinate*"

(iv) $\overline{\langle v, u\rangle} = \langle u, v\rangle$ "*Hermitian*"

(v) $\langle u, u\rangle > 0$ if $u \neq 0$, "*positive-definite*"

for all $u, v, w \in X$ and $k \in \mathbb{C}$.

Conditions (i)–(iii) are called sesquilinear. [Latin prefix *sesqui-* means "one and a half".]

A map satisfying (i)–(iv) is called a symmetric sesquilinear, or complex Hermitian form.

# Complex Fourier series

Let $X$ and $U$ be a complex inner product spaces.

For any vectors $x$ and $y$,

$$||x+y||^2 = ||x||^2 + \langle x, y\rangle + \langle y, x\rangle + ||y||^2 = ||x||^2 + 2\Re\langle x, y\rangle + ||y||^2.$$

Most results for real spaces carry over to complex spaces; just replace $T$ with $H$.

The adjoint of a linear map $A\colon X \to U$ is the map $A^*\colon U \to X$ such that

$$\langle x, A^* u\rangle = \langle Ax, u\rangle, \qquad \forall x \in X,\ u \in U.$$

## Proposition

The adjoint of $A\colon X \to U$ is its conjugate transpose, $A^* = A^H := \overline{A}^T$.

Two vectors $x, y$ are orthogonal if $\langle x, y\rangle = 0$. The vectors $x_1, \dots, x_k$ in $X$ are orthonormal if

$$\langle x_i, x_j\rangle = x_j^H x_i = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

## Unitary maps

Recall that an isometry of a real inner product space fixing 0 is called orthogonal.

An isometry of a complex inner product space fixing 0 is called unitary.

The matrix $A$ is orthogonal if $A^T A = I$, and unitary if $A^H A = I$.

Note that

- orthogonal means $A^* = A^{-1}$ in an $\mathbb{R}$-vector space
- unitary means $A^* = A^{-1}$ in a $\mathbb{C}$-vector space.

### Proposition

Let $U\colon X \to X$ be unitary.

(i) $U$ is linear

(ii) $U^* U = I$ (and conversely)

(iii) $U$ is invertible, and $U^{-1}$ is an isometry

(iv) $|\det U| = 1$.

The unitary maps form the unitary group, denoted $U(n)$ or $U_n$. The special unitary group $SU(n)$ are those with determinant 1.

# Complex Fourier series

Consider the space $X = \text{Per}_{2\pi}(\mathbb{C})$ of $2\pi$-periodic complex-valued functions.

We can define an inner product as

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\overline{g(x)}\, dx.$$

The set

$$\left\{ e^{inx} \mid n \in \mathbb{Z} \right\} = \left\{ \dots, e^{-2ix}, e^{-ix}, 1, e^{ix}, e^{2ix}, \dots \right\}$$

is an orthonormal basis w.r.t. to this inner product.

Thus, we can write each $f(x) \in \text{Per}_{2\pi}$ *uniquely* as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} = c_0 + \sum_{n=1}^{\infty} c_n e^{inx} + c_{-n} e^{-inx},$$

where

$$c_n = \text{proj}_{e^{inx}}\left(f\right) = \langle f, e^{inx} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx.$$