Lecture 6.3: Normal linear maps

Matthew Macauley

School of Mathematical & Statistical Sciences
Clemson University
http://www.math.clemson.edu/~macaule/

Math 8530, Advanced Linear Algebra

Commuting self-adjoint maps

When we studied Jordan canonical form, we proved the following

Corollary 4.14

Let $A, B\colon X \to X$ be commuting diagonalizable linear maps. Then they are simultaneously diagonalizable. That is, for some invertible $P\colon X \to X$,

$$A = PD_AP^{-1} \qquad \text{and} \qquad B = PD_BP^{-1}.$$

This is *almost* enough to establish the following:

Theorem 6.4

Suppose H and K are self-adjoint commuting maps. Then they have a common spectral resolution. That is, there are orthogonal projections $P_j\colon X \to X$ such that

$$I = \sum_{j=1}^{k} P_j, \qquad H = \sum_{j=1}^{k} \lambda_j P_j, \qquad K = \sum_{j=1}^{k} \mu_j P_j$$

Proposition 6.5

Let $A\colon X \to X$ be an anti-self-adjoint map of an inner product space. Then

(i) the eigenvalues of A are purely imaginary,

(ii) X has an orthonormal basis of eigenvectors of A.

Which maps have orthonormal eigenvectors?

Notice that the following linear maps all have orthonormal bases of eigenvectors:

1. self-adjoint: $H^* = H$
2. anti-self-adjoint: $A^* = -A$
3. orthogonal: $Q^* = Q^T = Q^{-1}$
4. unitary: $U^* = \overline{U}^T = U^{-1}$

The following generalizes all of these:

Definition

A linear map $N \colon X \to X$ is normal if $N^*N = NN^*$.

Note that NN^* and N^*N are self-adjoint, and hence normal.

Theorem 6.6

If $N \colon X \to X$ is normal, then X has an orthonormal basis of eigenvectors of N.

The reason *why* this holds is because $N = \dfrac{N + N^*}{2} + \dfrac{N - N^*}{2} = H + A$.

Properties of normal linear maps

Proposition 6.7

For a linear map $M\colon X \to X$ on an inner product space,

(i) if $\langle Mx, x\rangle = 0$ for all $x \in X$, then $M = 0$.

(ii) M is normal if and only if

$$||Mx|| = ||M^* x||, \qquad \text{for all } x \in X.$$

Corollary 6.8

If $N\colon X \to X$ is normal, then N and N^* have the same nullspace.

Unitary linear maps

Proposition 6.9

Let $U\colon X \to X$ be unitary. Then

1. X has an orthonormal basis of eigenvectors
2. each eigenvalue has norm 1.