Lecture 6.4: The Rayleigh quotient

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Math 8530, Advanced Linear Algebra

Overview

We derived the spectral resolution of self-adjoint maps using the spectral theory of linear maps.

In this lecture, we'll give an alternate proof that has several advantages:

1. It doesn't assume the fundamental theorem of algebra.
2. Over $\mathbb{R}$, it avoids complex numbers.
3. It leads to a "min-max principle" which characterizes eigenvalues and eigenvectors as critical points of a particular function.

Throughout, let $H\colon X \to X$ be self-adjoint, with

- eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$
- orthonormal eigenvectors $v_1, \dots, v_n$.

Recall that

$$\langle x, x\rangle = \sum_{j=1}^{n} a_j^2 \qquad \text{and} \qquad \langle x, Hx\rangle = \sum_{j=1}^{n} \lambda_j a_j^2.$$

The Rayleigh quotient

Definition

For a self-adjoint map $H\colon X \to X$, define the Rayleigh quotient of H as

$$R\colon X \setminus \{0\} \longrightarrow \mathbb{R}, \qquad R(x) = R_H(x) = \frac{\langle x, Hx \rangle}{\langle x, x \rangle} = \Big\langle \tfrac{x}{||x||},\ H\tfrac{x}{||x||} \Big\rangle.$$

Note that if $Hv_i = \lambda_i v_i$, then $R(v_i) = \lambda_i$.

Goal

Show that the critical points occur at the eigenvectors of H, and deduce that H has a full set of eigenvectors.

The Rayleigh quotient's minimum value

Since $R(x) = \dfrac{\langle x, Hx \rangle}{\langle x, x \rangle} = R(kx)$, we can think of R as being a map from the unit sphere.

This is compact (closed and bounded), so $R(x)$ achieves a minimum and maximum value.

Let $v \in X$ satisfy $R(v) = \min\limits_{||u||=1} R(u) := \lambda$.

Goal

Show that $Hv = \lambda v$, and that λ is the smallest eigenvalue of H.

Pick any other vector $w \in X$, a parameter $t \in \mathbb{R}$, and consider $R(v + tw)$.

The second-smallest eigenvalue of H

Let $v_1 \in X$ satisfy $R(v_1) = \min_{||u||=1} R(u) := \lambda_1$.

We just showed that $Hv_1 = \lambda_1 v_1$, and λ_1 is the smallest eigenvalue.

Now, let

$$X_1 := \mathrm{Span}(v_1)^\perp, \qquad \text{and so} \qquad X = X_1 \oplus \mathrm{Span}(v_1), \qquad \dim X_1 = n-1.$$

Goal

(i) Show that X_1 is H-invariant

(ii) Repeat the previous step (minimize the Rayleigh quotient) on X_1

(iii) Define $X_2 = \mathrm{Span}(\{v_1, v_2\})^\perp$, and iterate this process.

The min-max principle

Theorem 6.8

Let $H\colon X \to X$ be self-adjoint with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$. Then

$$\lambda_k = \min_{\dim S = k} \left\{ \max_{x \in S \setminus 0} R_H(x) \right\}.$$

Summary and applications of the Rayleigh quotient

For a self-adjoint map $H\colon X \to X$, the Rayleigh quotient of H is

$$R\colon X \setminus \{0\} \longrightarrow \mathbb{R}, \qquad R(x) = R_H(x) = \frac{\langle x, Hx\rangle}{\langle x, x\rangle} = \Big\langle \tfrac{x}{||x||},\ H\tfrac{x}{||x||}\Big\rangle.$$

Summary of the Rayleigh quotient

(i) The eigenvectors of H are the critical points of $R_H(x)$, i.e., the first derivatives of $R_H(x)$ are zero iff x is an eigenvector.

(ii) $R_H(v_i) = \lambda_i$ for any $Hv_i = \lambda_i v_i$.

(iii) In particular,

$$\lambda_1 = \min_{x\neq 0} R_H(x), \qquad \lambda_n = \max_{x\neq 0} R_H(x).$$

Application to numerical linear algebra

Let H be real-symmetric with $Hv = \lambda v$. If $||v - w|| \leq \epsilon$, then $|\lambda - R_H(w)| \leq \mathcal{O}(\epsilon^2)$.

That is, $R_H(w)$ is a 2nd order Taylor approximation of the eigenvalue.