Lecture 7.1: Definiteness and indefiniteness

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Basic concepts, and relation to eigenvalues

Definition

A self-adjoint map $M\colon X \to X$ is positive-definite (or positive) if

$$(x, Mx) > 0, \qquad \text{for all } x \neq 0,$$

and positive semi-definite (or nonnegative) if

$$(x, Mx) \geq 0, \qquad \text{for all } x \neq 0,$$

We denote these as $M > 0$ and $M \geq 0$, respectively.

Proposition 7.1

A self-adjoint map $M\colon X \to X$ is

(i) positive if and only if all eigenvalues of M are positive,

(ii) non-negative if and only if all eigenvalues of M are nonnegative.

We can define what it means for M to be negative, or non-positive, analogously.

A matrix that is none of these is said to be indefinite.

Basic properties of positive maps

Proposition 7.2

Let X be an inner product space, and $M, N, Q \in \text{Hom}(X, X)$.

(i) If $M, N > 0$, then $M + N > 0$ and $aM > 0$ for $a > 0$.

(ii) If $M > 0$ and Q invertible, then $Q^* MQ > 0$.

(iii) Every positive map has a unique positive square root.

The topology of positive maps

In an inner product space (or any metric space), the ball of radius $r > 0$ centered at $x \in X$ is

$$B_r(x) = \{y \in X : ||x - y|| < r\}.$$

Let $U \subseteq X$ be a subset. Then

- a point $u \in U$ is interior if there is some $\epsilon > 0$ for which $B_r(u) \subseteq U$.
- the set U is open if every $u \in U$ is interior
- the boundary of U is all $u \in U$ that are not interior.

Proposition 7.3

Let X be an inner product space, and consider the vector space of self-adjoint maps of X.

(i) The subset of positive maps is open.

(ii) The boundary of subset of positive maps are the non-negative maps.