Lecture 7.2: Nonstandard inner products

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Inner products and positive maps

Let X be a vector space with inner product $(\cdot\,,\cdot)$.

A positive map $M > 0$ defines a nonstandard inner product $\langle\cdot\,,\cdot\rangle$, where

$$\langle x, y\rangle := (x, My).$$

Lemma (HW)

If $H, M\colon X \to X$ are self-adjoint and $M > 0$, then $M^{-1}H$ is self-adjoint with respect to the inner product $\langle x, y\rangle = (x, My)$.

Definition

If $H, M\colon X \to X$ are self-adjoint and $M > 0$, the generalized Rayleigh quotient is

$$R_{H,M}(x) = \frac{(x, Hx)}{(x, Mx)} = \frac{(x, MM^{-1}Hx)}{(x, Mx)} = \frac{\langle x, M^{-1}Hx\rangle}{\langle x, x\rangle} := R_{M^{-1}H}\langle x\rangle \quad \text{w.r.t. } \langle\,,\rangle.$$

Note that:

- the ordinary Rayleigh quotient is simply $R_H = R_{H,I}$.
- the generalized Rayleigh quotient is an ordinary Rayley quotient.

The generalized Rayleigh quotient

Key remark

Results on the generalized Rayleigh quotient $R_{H,M}(x)$ follow from interpreting results of the ordinary Rayleigh quotient to

$$R_{M^{-1}H}\langle x\rangle := \frac{\langle x, M^{-1}Hx\rangle}{\langle x,x\rangle} = \frac{(x,Hx)}{(x,Mx)} = R_{H,M}(x).$$

For example, the minimum value of the Rayleigh quotient is the smallest eigenvalue of H:

$$R_H(v_1) = \lambda_1, \qquad \text{where } Hv_1 = \lambda_1 v_1.$$

The minimum value of the generalized Rayleigh quotient is the smallest eigenvalue of $M^{-1}H$:

$$R_{H,M}(v_1) = R_{M^{-1}H}\langle w_1\rangle = \mu_1, \qquad \text{where } M^{-1}Hw_1 = \mu_1 w_1.$$

Now, w.r.t. the inner product $\langle\, ,\rangle$, let

$$X_1 := \operatorname{Span}(v_1)^{\perp}, \qquad \text{and so} \qquad X = X_1 \oplus \operatorname{Span}(v_1), \qquad \dim X_1 = n-1.$$

The minimum value of the generalized Rayleigh quotient on X_1 is

$$\mu_2 = \min_{||x||=1} \left\{ R_{M^{-1}H}\langle x\rangle \mid \langle x, M^{-1}Hv\rangle = 0 \right\} = \min_{||x|||=1} \left\{ R_{H,M}(x) \mid (x, Mv) = 0 \right\}$$

where $M^{-1}Hw_2 = \mu_1 w_2$, and μ_2 is the 2nd smallest eigenvalue of $M^{-1}H$.

The min-max principle for the generalized Rayleigh quotient

Theorem 6.8

Let $H\colon X \to X$ be self-adjoint with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$. Then

$$\lambda_k = \min_{\dim S = k} \left\{ \max_{x \in S \setminus 0} R_H(x) \right\}.$$

Proposition (HW)

Let $H, M\colon X \to X$ be self-adjoint and $M > 0$.

1. Show that there exists a basis $v_1, \ldots, v_n$ of X where each v_i satisfies

$$Hv_i = \mu_i M v_i \quad (\mu_i \text{ real}), \qquad (v_i, Mv_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

2. Compute (v_i, Hv_j), and show that there is an invertible matrix U for which $U^* M U = I$ and $U^* H U$ is diagonal.
3. Characterize the numbers $\mu_1, \ldots \mu_n$ by a minimax principle.