Lecture 7.3: Gram matrices

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The matrix $A^T A$

Consider an $n \times m$ matrix A over $\mathbb{R}$, where

$$A = \begin{bmatrix} x_1 & \cdots & x_m \end{bmatrix}$$

The $m \times m$ matrix $A^T A$ is self-adjoint:

$$A^T A = \begin{bmatrix} x_1^T x_1 & x_1^T x_2 & \cdots & x_1^T x_m \\ x_2^T x_1 & x_2^T x_2 & \cdots & x_2^T x_m \\ \vdots & \vdots & \ddots & \vdots \\ x_m^T x_1 & x_m^T x_2 & \cdots & x_m^T x_m \end{bmatrix}$$

Note that $A \colon \mathbb{R}^m \to \mathbb{R}^n$ and $A^T A \colon \mathbb{R}^m \to \mathbb{R}^m$. We've already seen that:

1. $\operatorname{rank} A = \operatorname{rank} A^T A$ and $\operatorname{nullity} A = \operatorname{nullity} A^T A$ (in fact, $N_A = N_{A^T A}$)
2. $A^T A \geq 0$
3. If $N_A = 0$, then the projection matrix onto $\operatorname{Span}(x_1, \ldots, x_m)$ is $A(A^T A)^{-1} A^T$.

This is an example of a Gram matrix.

Later, we'll diagonalize $A^T A$ to get the celebrated singular value decomposition of A.

The matrix A^*A

Now, we'll generalize the construction A^TA with the standard inner product to A^*A to an arbitrary inner product.

We'll see how the construct *all* positive matrices.

Definition

Let $x_1, \dots, x_m \in X$, with inner product $(\ ,\ )$. The Gram matrix of these vectors is

$$G = (G_{ij}), \qquad \text{where} \quad G_{i,j} = (f_i, f_j).$$

Notice that $G = A^*A$, where $A = \begin{bmatrix} x_1 & \cdots & x_m \end{bmatrix}$.

Theorem 7.4

1. Every Gram matrix is nonnegative.
2. The Gram matrix of a set of linearly independent vectors is positive.
3. Every positive matrix is a Gram matrix.

Examples

1. Let $X = \{f \colon [0,1] \to \mathbb{R}\}$, where $(f,g) = \int_0^1 f(t)g(t)\,dt$. If

$$f_1 = 0, \quad f_2 = t, \quad \dots\,, \quad f_i = t^{i-1},$$

then the Gram matrix is $G = (G_{ij})$, where

$$G_{ij} = \frac{1}{i+j-1}.$$

2. Given a "weighting function" $w \colon \mathbb{R} \to \mathbb{R}^+$, define

$$(f,g) = \int_0^{2\pi} f(\theta)\overline{g(\theta)}w(\theta)\,d\theta.$$

If $f_j = m$, then the $(2n+1) \times (2n+1)$ Gram matrix is $G_{kj} = c_{k-j}$, where

$$c_p = \int w(\theta)e^{-ip\theta}\,d\theta.$$

The entry-wise product of matrices

Let $A = (a_{ij})$ and $B = (b_{ij})$ be matrices of the same size. Then define

$$A \circ B = (a_{ij} b_{ij}).$$

Schur's product theorem

If $A, B > 0$, then so is $A \circ B$.