Lecture 7.6: The partial order of positive maps

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Partially ordered sets

Recall that a partial order on a set X is a relation $\leq$ that is:

(i) reflexive: $x \leq x$

(ii) anti-symmetric: $x \leq y$ and $y \leq x \;\Rightarrow\; x = y$

(iii) transitive: $x \leq y \leq z \;\Rightarrow\; x \leq z$.

We say that $x < y$ if $x \leq y$ and $x \neq y$. The pair $(X, \leq)$ is a partially ordered set (poset).

Alternatively, we can define a partial order by a relation $<$ that is

(i) reflexive: $x \not\leq x$

(ii) anti-symmetric: $x < y \;\Rightarrow\; y \not< x$

(iii) transitive: $x < y < z \;\Rightarrow\; x < z$.

Definition

Put a following partial order on the set of self-adjoint maps:

$$M < N \quad \text{iff} \quad N - M > 0, \qquad\qquad M \leq N \quad \text{iff} \quad N - M \geq 0.$$

Basic properties of the poset of positive maps

The following easy facts all hold for positive numbers:

(i) If $m_1 < n_1$ and $m_2 < n_2$, then $m_1 + m_2 < n_1 + n_2$.

(ii) If $\ell < m < n$, then $\ell < n$.

(iii) If $m < n$ and $a > 0$, then $am < an$

(iv) If $0 < m < n$, then $1/m > 1/n > 0$.

Proposition 7.6

The following all hold for linear maps on X:

(i) If $M_1 < M_1$ and $M_2 < N_2$, then $M_1 + M_2 < N_1 + N_2$.

(ii) If $L < M < N$, then $L < N$.

(iii) Given maps $M < N$ and a scalar $a > 0$, we have $aM < aN$.

(iv) If $0 < M < N$, then $M^{-1} > N^{-1} > 0$.

The symmetrized product

Definition

If $A, B \colon X \to X$ are self-adjoint, their symmetrized product is

$$S = AB + BA.$$

Proposition 7.7

Let A, B be self-adjoint. If $A > 0$ and $AB + BA > 0$, then $B > 0$.