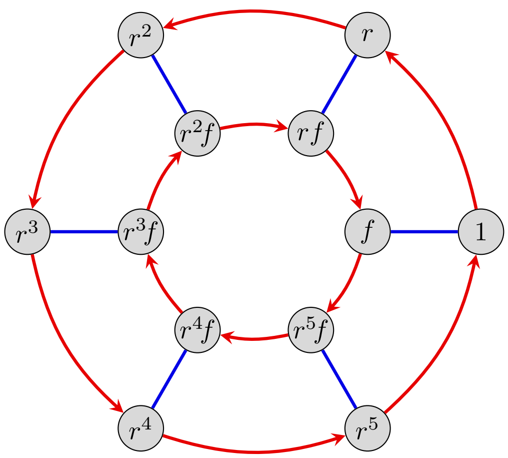
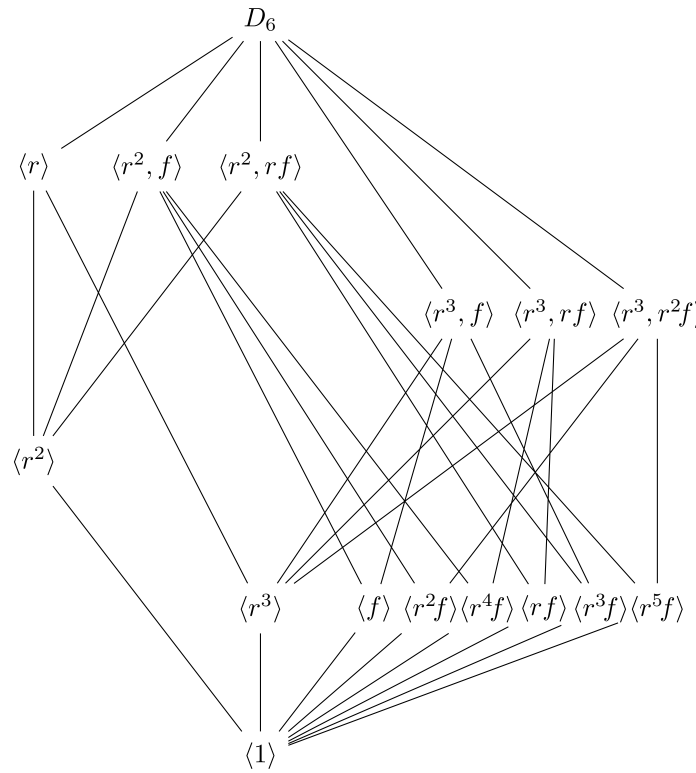
Math 4120, Midterm 2. Wednesday November 3, 2021

1. (35 points) Let $G = D_6$. A Cayley diagram and subgroup lattice are shown below.





(a) Which subgroups of D_6 are normal? Which one is the center, $Z(D_6)$?

(b) Write down the conjugacy classes of subgroups of D_6 that have size *larger* than 1.

(c) Draw the subgroup lattice of the quotient $D_6/\langle r^3\rangle$. What familiar group is this isomorphic to?

(d) Find the commutator subgroup D_6'. What familiar group is the abelianization D_6'/D_6 isomorphic to?

(e) Write D_6 as the semidirect product of two (nontrivial) subgroups, in as many ways as possible, up to isomoprhism. Justify your answer.

(f) Is D_6 isomorphic to the direct product of two (nontrivial) subgroups? Why or why not?

(g) Write down all (distinct) inner automorphisms of D_6. Denote $x \mapsto gxg^{-1}$ by φ_g. What familiar group is $\mathrm{Inn}(D_6)$ isomorphic to? [*Hint*: Recall that $\mathrm{Inn}(G) \cong G/Z(G)$.]

2. (15 points) Let H be a subgroup of an abelian group G.

(a) Show that H is abelian.

(b) Show that G/H is abelian.

3. (15 points) We have already seen examples, both in subgroup lattices and from old homework, of how dicyclic groups have an order-2 subgroup whose quotient yields a dihedral group. In this problem, you will establish this for all n. Define the map

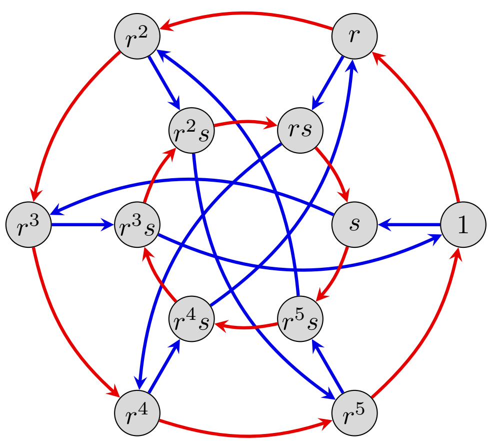
$$\varphi\colon \operatorname{Dic}_{2n} \longrightarrow D_n, \qquad \varphi(r^i s^j) = r^{i \bmod n} f^j.$$

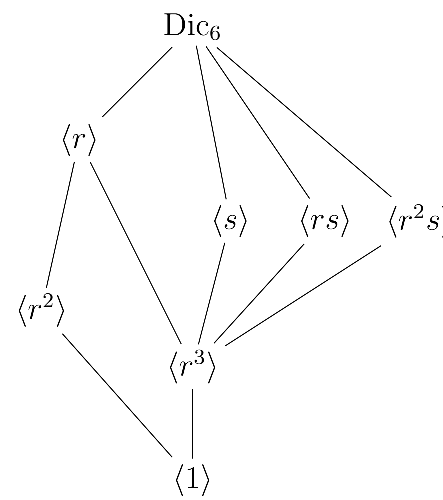
   (a) Show that φ is a homomorphism, and find $\operatorname{Ker}(\varphi)$.

   (b) Is this map 1-to-1? Is it onto? Justify your answers.

   (c) Show that $\operatorname{Dic}_{2n}/\langle r^n \rangle \cong D_n$.

   They aren't needed, but in case it helps, a Cayley diagram and subgroup lattice of Dic_6 are shown below.





4. (20 points) Let $H, N \leq G$ and suppose that $N \trianglelefteq G$. Show that

$$H/(H \cap N) \cong HN/N.$$

   You may assume that $HN \leq G$, and that both $N \trianglelefteq HN$ and $H \cap N \trianglelefteq H$. [*Hint*: Start with a map φ from H. Make sure you write down how it's defined.]

5. (15 points) We've seen what it means for muliplication of cosets in G/N to be *well-defined*. We've also seen what it means for a map $f\colon G/N \to H$ to be well-defined.

   (a) In plain English, in a single sentence, describe informally what "well-defined means" intuitively.

   (b) Write down a formal definition of what it means for coset multiplication in G/N to be well-defined.

   (c) Write down a formal definition of what it means for a map $f\colon G/N \to H$ to be well-defined.