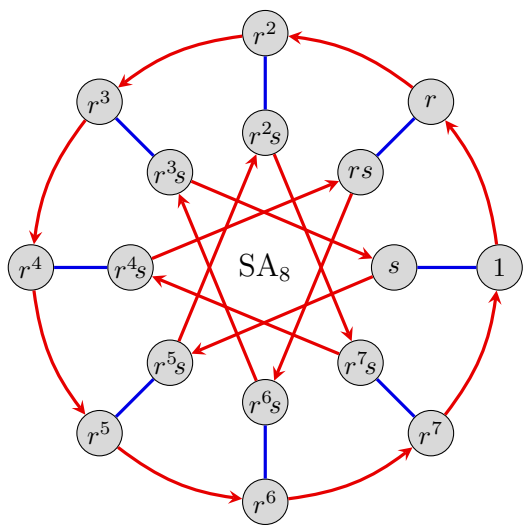
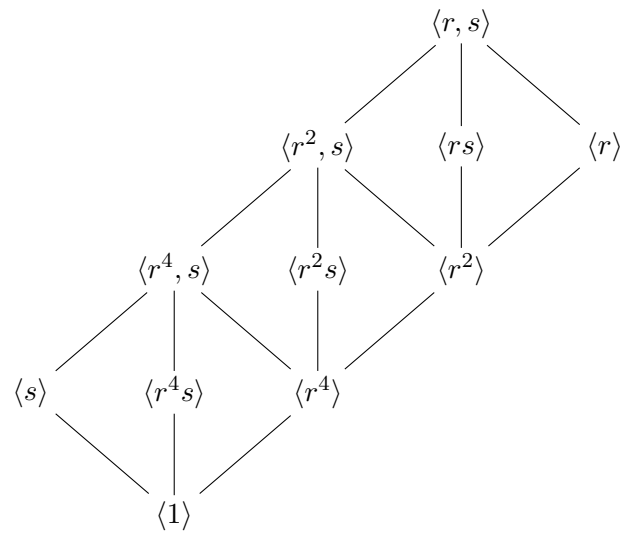
1. Let G be the *semiabelian group* of order 16, defined by the presentation

$$\mathrm{SA}_8 = \langle r, s \mid r^8 = s^2 = 1, srs = r^5 \rangle,$$

A Cayley diagram and subgroup lattice are shown below.





(a) The subgroups $V = \langle r^4, s\rangle$, $H = \langle r^2 s\rangle$, $K = \langle r^2\rangle$, and $N = \langle r^4\rangle$ are all normal. For the first three, highlight the cosets on a fresh Cayley diagram by colors.

(b) Construct a Cayley table for the quotient of G by each of these subgroups. Then draw a Cayley diagram for each, labeling the nodes with elements (i.e., cosets).

(c) Let $N = \langle r^4\rangle$. The shaded region below shows an order-4 cyclic subgroup of G/N, generated by the element rN, and how the union of these four cosets is the order-8 subgroup $\langle r\rangle$ of G. Construct analogous tables for the other five non-trivial proper subgroups of G/N, and then draw the subgroup lattice of G/N.

r^3N	r^3sN
r^2N	r^2sN
rN	rsN
N	sN

$\langle rN\rangle \leq G/N$

r^3	r^7	r^3s	r^7s
r^2	r^6	r^2s	r^6s
r	r^5	rs	r^5s
1	r^4	s	r^4s

$\langle r\rangle/N \leq G/N$

r^3	r^7	r^3s	r^7s
r^2	r^6	r^2s	r^6s
r	r^5	rs	r^5s
1	r^4	s	r^4s

$\langle r\rangle \leq G$

(d) For each subgroup H/N from Part (c), determine what the quotient of G/N by H/N is isomorphic to? Justify your answer.

(e) One step of Part (c) consisting of starting with G, taking the quotient by N, and then taking the subgroup generated by r^2N and sN. Compare and contrast this to doing these steps in the reverse order. That is: *start with G, first take the subgroup $\langle r^2, s\rangle$, and then take the quotient by N.*

(f) Repeat Part (c) for the subgroups $\langle r^4, s\rangle$, $\langle r^2 s\rangle$, and $\langle r^2\rangle$ of G.

2. Let H be a subgroup of G.

   (a) Show that if G is abelian, then H and G/H are abelian.

   (b) Show that if $G/Z(G)$ is cyclic, then G is abelian.

   (c) What cyclic groups can arise as a quotient $G/Z(G)$? Justify your answer.

3. Let X be a subset of a group G. The *centralizer* of X, denoted $C_G(X)$, is the set of all elements that commute with everything in X:

$$C_G(X) = \big\{g \in G \mid gx = xg,\ \forall x \in X\big\}.$$

   If $X = \{x\}$, then we denote the centralizer as $C_G(x)$.

   (a) Show that $C_G(X)$ is a subgroup of G.

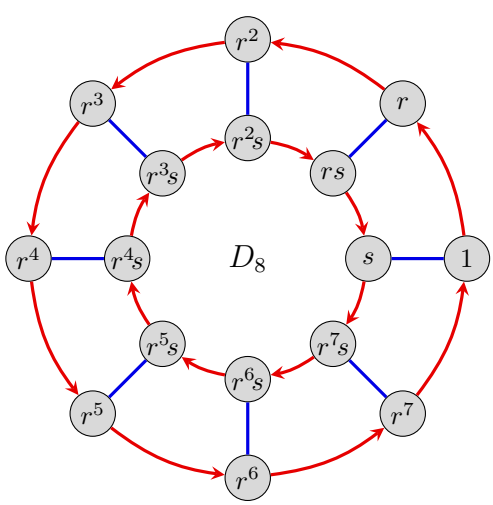
   (b) If H is a subgroup of G, show that $C_G(H) \trianglelefteq N_G(H)$.

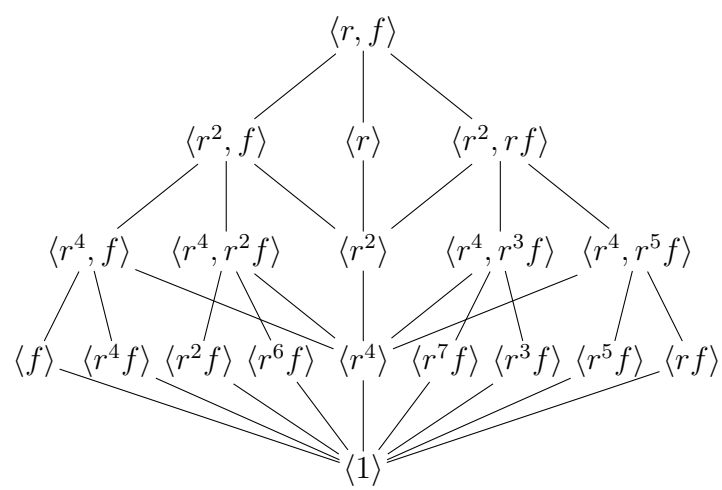
   (c) For Q_8 and D_6, compute the centralizers of each element $x \in G$, as well as $N_G(\langle x\rangle)$. What pattern do you notice about the size of $C_G(x)$ vs. $\mathrm{cl}_G(x)$? The partition of these groups by conjugacy classes is shown below.

1	i	j	k
-1	$-i$	$-j$	$-k$

1	r	r^2	f	r^2f	r^4f
r^3	r^5	r^4	rf	r^3f	r^5f

   (d) Partition the elements of the groups D_5 and D_8 by conjugacy classes, and arrange them in a table, as above. Then repeat the previous part for these groups. The Cayley diagram and subgroup lattice for D_8 is shown below, for convenience.





4. Recall that two elements in S_n are conjugate if and only if they have the same cycle type.

   (a) Partition the elements of S_4 by conjugacy class. How many are in each class?

   (b) Compute the centralizers of e, (12), (123), (1234), and $(12)(34)$ in S_4. What do you notice about the size of $C_G(x)$ vs. $\mathrm{cl}_G(x)$?

   (c) Partition the elements of A_4 by conjugacy class. Then pick one element from each class, and find its centralizer.

   (d) For each of the elements e, (12), (123), (1234), (12345), $(12)(34)$, and $(123)(45)$ in S_5, determine the size of its conjugacy class, and then find its centralizer.