1. Let $A, B \leq G$ and say that A normalizes B (that is, $A \leq N_G(B)$).
   (a) Show that AB is a subgroup of G.
   (b) Show that $B \trianglelefteq AB$ and $A \cap B \trianglelefteq A$.
   (c) Show that $A/(A \cap B) \cong AB/B$. [*Hint:* Construct a homomorphism $\phi\colon A \to AB/B$ that has $A \cap B$ as kernel, then apply the FHT.]

2. Recall that the *commutator subgroup* is defined as

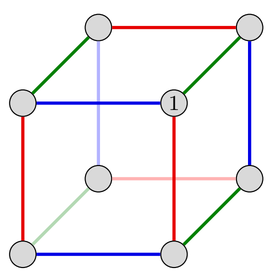
$$G' = \big\langle xyx^{-1}y^{-1} \mid x, y \in G \big\rangle.$$

   (a) Show that G' is the intersection of all normal subgroups of G that contain the set $C := \{aba^{-1}b^{-1} \mid a, b \in G\}$:

$$G' = \bigcap_{C \subseteq N \trianglelefteq G} N.$$

   (b) Show that G/G' is abelian.
   (c) For each of the groups SD_8, $\mathrm{AGL}_1(\mathbb{Z}_5)$, Dic_{10}, and $\mathrm{SL}_2(\mathbb{Z}_3)$, whose subgroup lattices are shown on the supplemental material, carry out the following steps. Because all of this can be done by inspection, you need to briefly justify your answers.
      (i) Partition the subgroups into conjugacy classes, by drawing dashed circles on the lattices.
      (ii) The *derived series* of a group is defined as $G^{(0)} := G$, $G^{(1)} := G'$, and inductively, $G^{(k)}$ is the commutator of $G^{(k-1)}$. Mark these groups on the lattice until the trivial group is reached, and determine the quotient $G^{(i+1)}/G^(i)$ of each successive pair.

3. The Cayley diagram shown below describes a group G that is a semidirect product of V_4 with C_2. In this problem, we will break down the construction of this into steps.



   (a) Label each node on this Cayley diagram with the order of the corresponding element. Just from this information alone, explain what familiar group this is, and why.
   (b) On a fresh copy of this Cayley diagram, label the nodes with elements of this group.
   (c) Recall that $\mathrm{Aut}(V_4) = \langle \alpha, \beta \rangle \cong S_3 \cong D_3$, where

$$\boldsymbol{\alpha}: \quad h \quad v \quad r \qquad\qquad \boldsymbol{\beta}: \quad h \quad v \quad r$$

Let $C_2 = \langle s \rangle = \{1, s\}$. Construct the group G as a semidirect product

$$G \cong V_4 \rtimes_\theta C_2 = \big\{(a,b) \mid a \in V_4,\, b \in C_2\big\}.$$

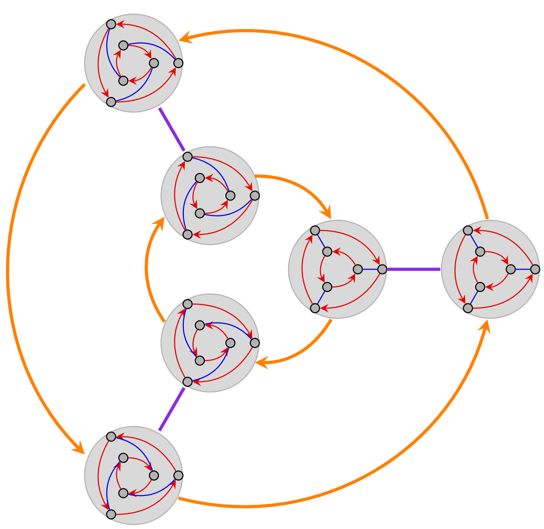
That is, define an explicit labeling map $\theta\colon C_2 \to \mathrm{Aut}(V_4)$. [*Note*: There's more than one map that will work.]

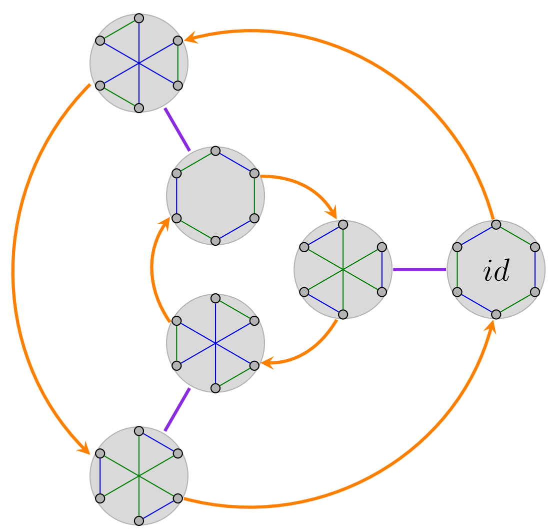
(d) On a fresh copy of the Cayley diagram, label each node as an ordered pair in $V_4 \rtimes_\theta C_2$.

4. Recall that the automorphism group of D_3 is $\mathrm{Aut}(D_3) = \langle \alpha, \beta \rangle \cong D_3$, where

$$\begin{cases} \alpha(r) = r \\ \alpha(f) = rf \end{cases} \qquad\qquad \begin{cases} \beta(r) = r^2 \\ \beta(f) = f \end{cases}$$

All of these automorphisms are inner, because $\mathrm{Inn}(D_3) \trianglelefteq \mathrm{Aut}(D_3)$ and $\mathrm{Inn}(D_3) \cong D/Z(D_3) \cong D_3$. Two Cayley diagrams for $\mathrm{Aut}(D_3)$ are shown below.





In this problem, we will analyze the automorphism group of D_4, and construct similar Cayley diagrams.

(a) The inner automorphism group of D_4 is

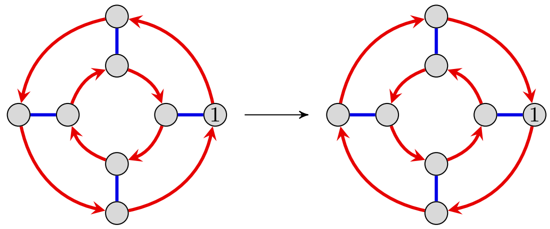
$$\mathrm{Inn}(D_4) = \big\{\, \mathrm{Id}, \varphi_r, \varphi_f, \varphi_{rf} \big\} \cong D_4/Z(D_4) = D_4/\langle r^2 \rangle \cong V_4,$$

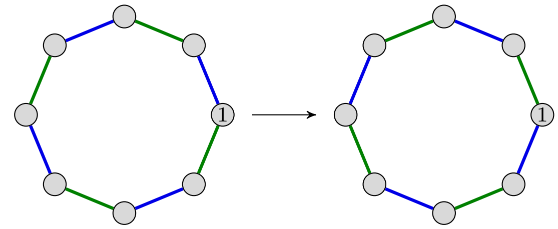
and additionally, there is a nontrivial outer automorphism

$$\alpha\colon D_4 \longrightarrow D_4, \qquad \alpha(r) = r, \quad \alpha(f) = rf.$$

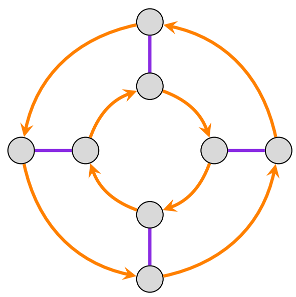
that swaps the "two types" of reflections of the square. Construct a Cayley diagram for $\mathrm{Aut}(D_4)$, as the semidirect product of $\mathrm{Inn}(D_4) = \langle \varphi_r, \varphi_f \rangle$ and $\mathrm{Out}(D_4) = \langle \alpha \rangle$. Label each node with the product of an element of $\mathrm{Inn}(D_4)$ and an element of $\mathrm{Out}(D_4)$. The previous problem should be very helpful.

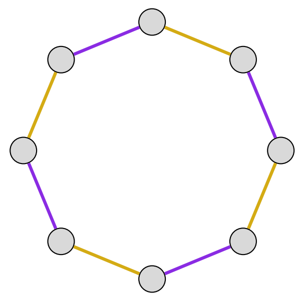
(b) Determine which elements from the previous part are the following automorphisms, and whether or not they are inner.

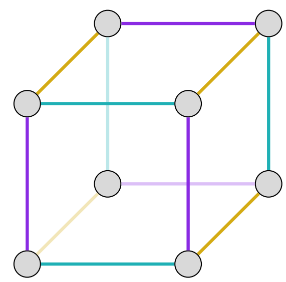




(c) As was done above for $\mathrm{Aut}(D_3)$, create the Cayley diagram for $\mathrm{Aut}(D_4)$, and label the nodes with re-wired copies of the Cayley diagram of $D_4 = \langle r, f \rangle$. Do this for each of the following Cayley diagrams of $\mathrm{Aut}(D_4)$ shown below.







(d) Repeat the previous part using the Cayley diagram of $D_4 = \langle s, t \rangle = \langle f, rf \rangle$.