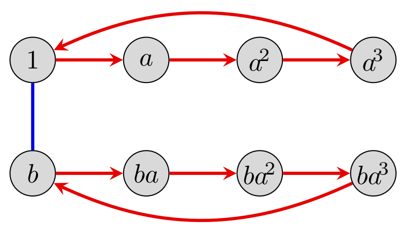
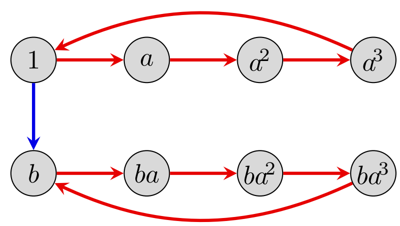
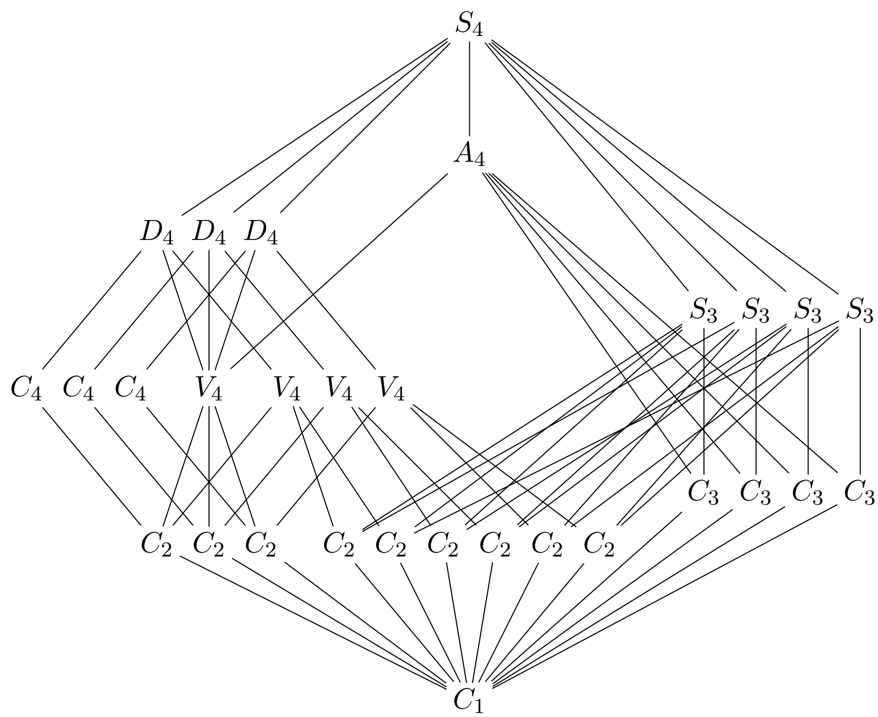
1. Let G be an unknown group of order 8. If it has no element of order 4, then $g^2 = e$ for all $g \in G$, and so G must be abelian. Otherwise, it has a "partial Cayley diagram" like one of the following:





Find all possibilities for finishing each diagram, and label by isomorphism type.

2. The subgroup lattice of the symmetric group S_4 is shown below.



   (a) Partition the subgroups into conjugacy classes. Carefully and completely justify your answers using the Sylow theorems, without making reference to cycle type.
   (b) For each conjugacy class $\mathrm{cl}_G(H)$, find the isomorphism type of the normalizer $N_G(H)$.
   (c) Using the GroupNames website, make a table of all 15 groups of order 24, the number of subgroups, and basic information about their Sylow p-subgroups (number and isomorphism type). Write down at least one observation that you find interesting.
   (d) Which groups are *not* an internal direct or semidirect product of Sylow subgroups?
   (e) None of the following groups are among the 15 listed on GroupNames: $D_6 \times C_2$, $C_6 \rtimes C_4$, $C_6 \rtimes C_2^2$, $C_4 \rtimes C_6$, $C_3 \rtimes C_2^3$, $C_2^3 \rtimes C_3$, $C_2^2 \rtimes C_6$, $C_3 \rtimes Q_8$, $Q_8 \rtimes C_3$, $C_4 \rtimes S_3$. Find which of the 15 each is isomorphic to, and add this this to your table.

3. Show that there are no simple groups of the following order.

   (i) $45 = 3^2 \cdot 5$ (ii) $56 = 2^3 \cdot 7$ (iii) $108 = 2^3 \cdot 3^2$ (iv) p^n.

   [*Hint*: For Part (d), first use a suitable group action to show that $|Z(G)| > 1$.]

4. After A_5, the next smallest nonabelian simple group is $G = \mathrm{GL}_3(\mathbb{Z}_2)$, the invertible 3×3 binary matrices. It has order $168 = 2^3 \cdot 3 \cdot 7$.

   (a) What do the Sylow theorems tell us about the possibilities for n_2, n_3, and n_7?

   (b) Show that G is isomorphic to a subgroup of A_8. [*Hint*: Let G act on its Sylow 7-subgroups by conjugation.]

5. Let G be an unknown group of order 90.

   (a) Using the Sylow theorems, find all possibilities for n_2, n_3, and n_5, where n_p is the number of Sylow p-subgroups of G.

   (b) Suppose that G has a nonnormal Sylow 5-subgroup. Show that there is a non-trivial homomorphism $\phi\colon G \to S_6$.

   (c) If $\phi\colon G \to A_6$, show that ϕ is not injective. You may assume that A_6 is simple.

   (d) Show that G is not simple.