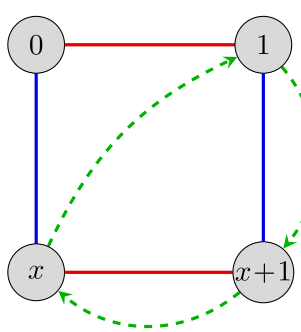
1. For each of the following rings R, determine the zero divisors and the set $U(R)$ of units.
   (a) The set $\mathcal{C}^1$ of continuous real-valued functions $f\colon \mathbb{R} \to \mathbb{R}$.
   (b) The polynomial ring $\mathbb{R}[x]$.
   (c) $\mathbb{Z} \times \mathbb{Z}$, where addition and multiplication are defined componentwise.
   (d) $\mathbb{R} \times \mathbb{R}$, where addition and multiplication are defined componentwise.

2. Let I and J be ideals of a ring R.
   (a) Show that if a left ideal I of a ring R contains a unit, then $I = R$.
   (b) Show that $I + J$, $I \cap J$, and IJ are ideals of R, where
   $$IJ = \big\{x_1y_1 + \cdots + x_ky_k \mid x_i \in I,\ y_j \in J\big\}.$$
   (c) If R is commutative, then the set
   $$(I : J) = \big\{r \in R \mid rJ \subseteq I\big\}$$
   is called the *colon ideal* of I and J. Show that $(I : J)$ is an ideal of R.
   (d) Consider the ideals $I = 4\mathbb{Z}$ and $J = 6\mathbb{Z}$ of the ring $R = \mathbb{Z}$. Compute $I + J$, $I \cap J$, IJ, $(I : J)$, and $(J : I)$.
   (e) Repeat Part (c) for the ideals $I = m\mathbb{Z}$ and $J = n\mathbb{Z}$ of $R = \mathbb{Z}$.

3. The finite field $\mathbb{F}_4$ on 4 elements can be constructed as the quotient of the polynomial $\mathbb{Z}_2[x]$ by the ideal $I = (x^2 + x + 1)$ generated by the irreducible polynomial $x^2 + x + 1$. The figure below shows a Cayley diagram, and multiplication and addition tables for the finite field $\mathbb{Z}_2[x]/(x^2 + x + 1) \cong \mathbb{F}_4$.



$+$	0	1	x	$x+1$
0	0	1	x	$x+1$
1	1	0	$x+1$	x
x	x	$x+1$	0	1
$x+1$	$x+1$	x	1	0

$\times$	1	x	$x+1$
1	1	x	$x+1$
x	x	$x+1$	1
$x+1$	$x+1$	1	x

The polynomials $f(x) = x^3 + x + 1 \in \mathbb{Z}_2[x]$ and $g(x) = x^2 + x + 2 \in \mathbb{Z}_3[x]$ are irreducible. Construct the Cayley tables and Cayley diagram for the finite fields
$$\mathbb{F}_8 \cong \mathbb{Z}_2[x]/(f) \qquad \text{and} \qquad \mathbb{F}_9 \cong \mathbb{Z}_3[x]/(g)\,.$$
What familiar groups appear as the additive and multiplicative groups of these fields?

4. The left ideal generated by $X \subseteq R$ is defined as

$$(X) := \bigcap \big\{ I \mid I \text{ is a left ideal s.t. } X \subseteq I \subseteq R \big\}.$$

   (a) Show that the left ideal generated by X is

$$(X) = \big\{ r_1 x_1 + \cdots + r_n x_n \mid n \in \mathbb{N},\ r_i \in R,\ x_i \in X \big\}.$$

   (b) The two-sided ideal generated by $X \subseteq R$ is defined by relacing "left" with "two-sided" in the definition above. Show that this is also equal to

$$\big\{ r_1 x_1 s_1 + \cdots + r_n x_n s_n \mid n \in \mathbb{N},\ r_i, s_i \in R,\ x_i \in X \big\}.$$

   (c) Find a (non-commutive) ring R and a set X such that the left and two-sided ideals generated by X are different.

5. Prove the Fundamental homomorphism theorem (FHT) for rings: If $\phi \colon R \to S$ is a ring homomorphism, then $\operatorname{Ker}(\phi)$ is a two-sided ideal of R, and $R/\operatorname{Ker}(\phi) \cong \operatorname{Im}(\phi)$. You may assume the FHT for groups.