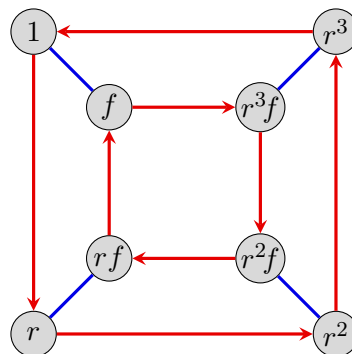
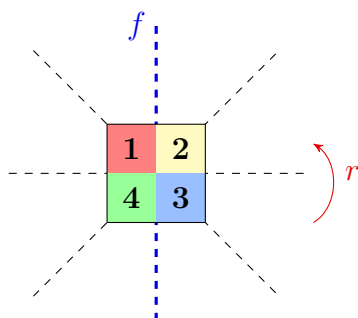
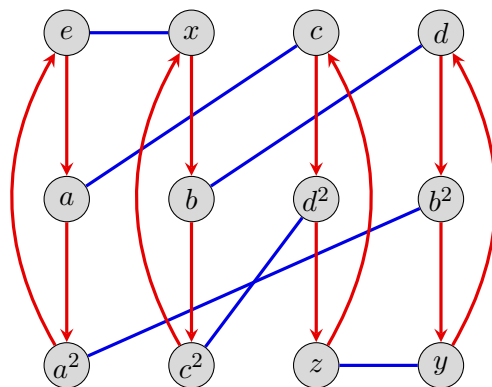
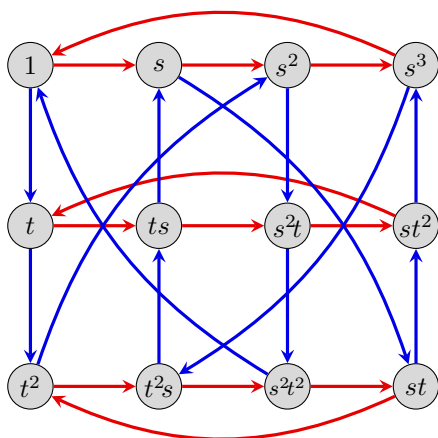


- The eight symmetries of a square form a group that we will call \mathbf{Sq} , generated by a 90° counterclockwise rotation r , and a horizontal flip f . A Cayley diagram is shown below.



- For each axis of reflection, express the symmetry across it in terms of r and f .
 - Find all *minimal* generating sets. [Hint: There are 12.]
 - Let $s = f$ and $t = r^3f = fr$. Draw a Cayley diagram using s and t as generators.
 - Write a presentation of the form $\mathbf{Sq} = \langle r, f \mid \dots \rangle$.
 - Write a presentation of the form $\mathbf{Sq} = \langle s, t \mid \dots \rangle$.
 - Construct a *Cayley table* for this group, ordered $1, r, r^2, r^3, f, rf, r^2f, r^3f$. Describe how the rotations and reflections are “clustered” in this table.
- The Cayley diagrams of two groups of size 12 are shown below.



- Create a Cayley table for each group. Order the elements in the first group by $1, t^2, s^2t^2, t, s^2, s^2t^2, s, st^2, t^2s, st, s^3, ts$ and those in second by $e, x, y, z, a, b, c, d, a^2, b^2, c^2, d^2$.
- Find the inverse of each element.
- Find the *order* of each g : the minimal $k > 0$ such that $g^k = e$, denoted $|g|$.
- Write a presentation for each group.
- Determine whether or not these two groups are isomorphic. Justify your answer.
- Squint your eyes. Do you see any patterns in these tables?

3. In this problem, we will define two variations of the \mathbf{Coin}_2 group from lecture. We will consider two types of tiles, and declare the following to be the “home state” of each:



Our first group is $\mathbf{Coin}_3 = \langle c, t \rangle$, where c “cyclically shifts” the entries, and t “toggles” the color of the leftmost square:

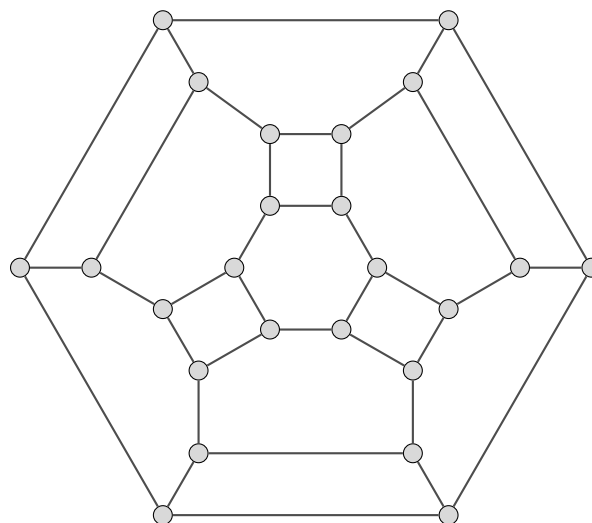
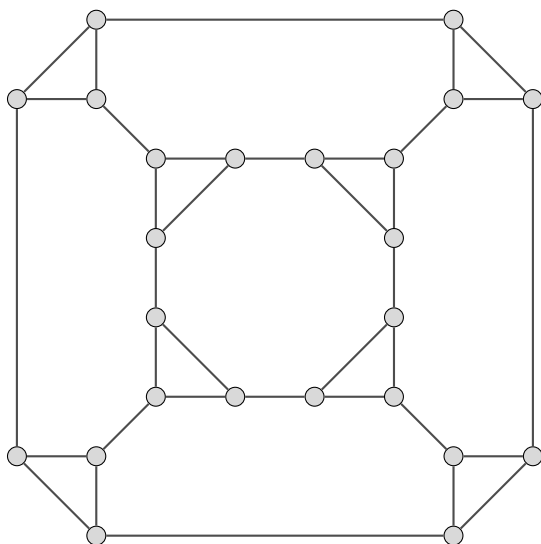


Our second group is $\mathbf{Box}_2 = \langle r, s \rangle$, where r “rotates” the squares counterclockwise, and s “swaps” the squares on the top row.



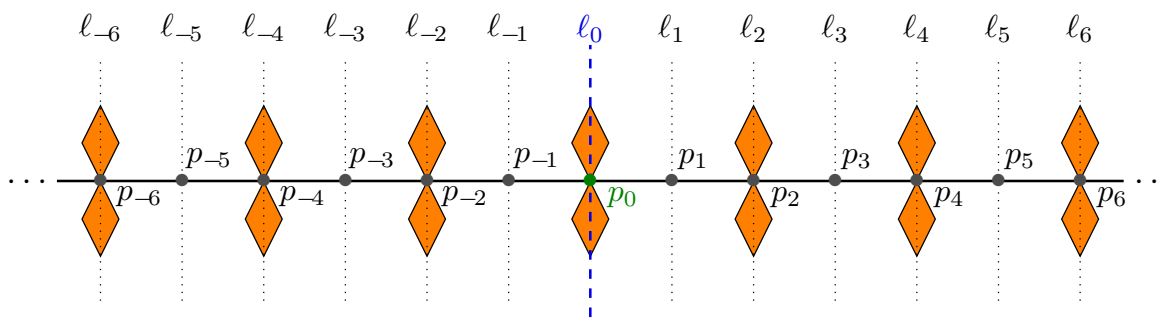
Note that the square tiles don’t actually need to be shaded. An alternate way to denote the colors of the 3×1 dominos is to underline any number with a black background. For example, using this convention, the “home state” would be written 1 2 3.

- (a) Both of these groups have 24 actions. Draw a Cayley diagram for each, with the nodes labeled by configurations. It is helpful to know that the one for \mathbf{Coin}_3 can be arranged on a *truncated cube*, whose skeleton is shown below (left). A Cayley diagram for \mathbf{Box}_2 can be arranged on a *truncated octahedron*, shown below (right).



- (b) Write down a presentation for each of these groups.
- (c) Are these groups isomorphic? Justify your answer.

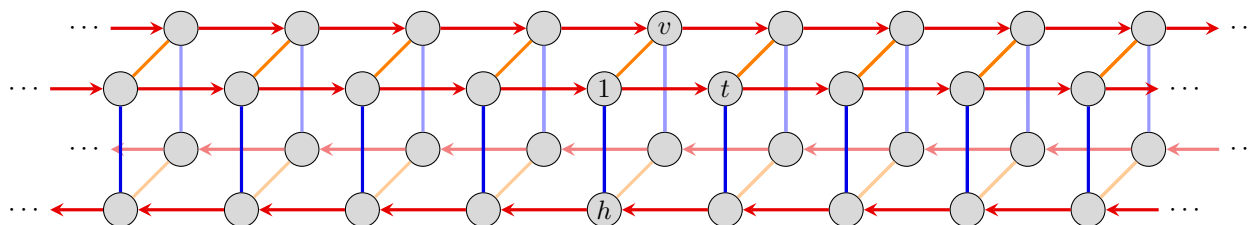
4. Consider the frieze shown below:



Let t be a minimal translation to the right, h_i a reflection across l_i , and r_j a 180° rotation around p_j . Let v be the vertical reflection and $g_i = t^i v$ a glide reflection right. A presentation for the frieze group is

$$\mathbf{Frz}_1 := \langle t, h, v \mid v^2 = h^2 = 1, th = ht^{-1}, tv = vt, hv = vh \rangle,$$

where $h = h_0$. A Cayley diagram is shown below.

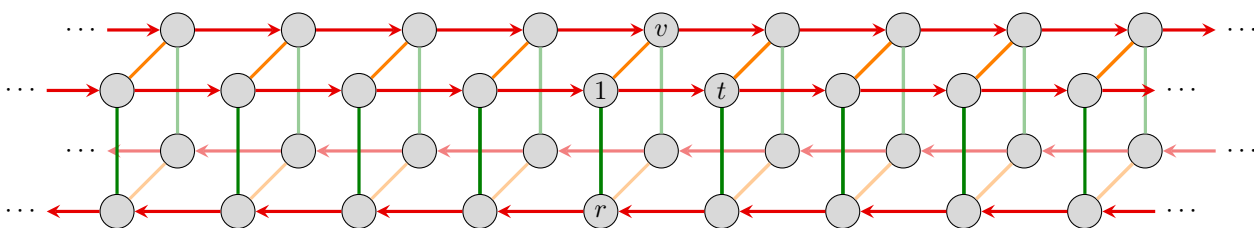


(a) Every symmetry is either a translation t^i , glide reflection g_j , rotation r_k , horizontal reflection h_ℓ , or the vertical reflection v . Label the vertices of this Cayley diagram with elements written in this form.

(b) Now, repeat the previous part, but with for the Cayley diagram for the presentation

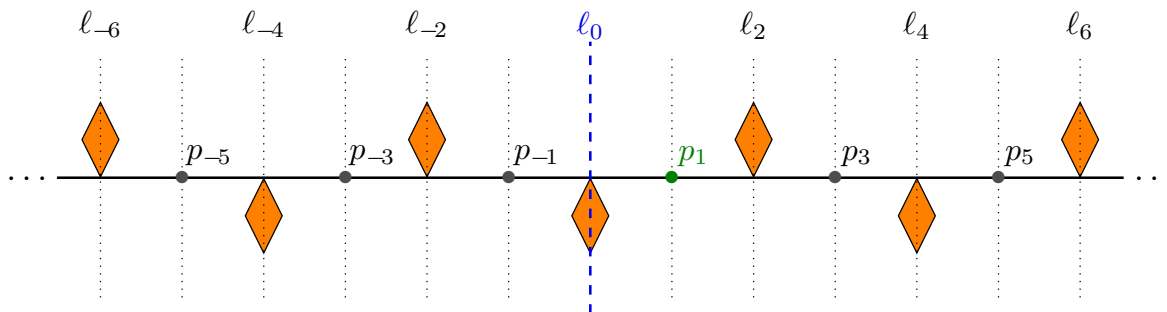
$$\mathbf{Frz}_1 := \langle t, r, v \mid v^2 = r^2 = 1, tr = rt^{-1}, tv = vt, rv = vr \rangle,$$

where $r = r_0$. A Cayley diagram is shown below.



(c) Determine which symmetries $t^i h t^{-i}$ and $t^i r t^{-i}$ are for each $i \in \mathbb{Z}$.

5. The subgroup $\mathbf{Frz}_2 = \langle g, h \rangle$ of the frieze group from the previous problem, where $g = g_1 = tv$, is the symmetry group of the following frieze:

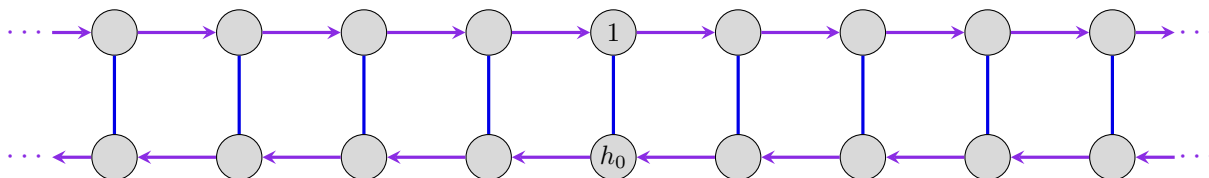


Two presentations for this frieze group are

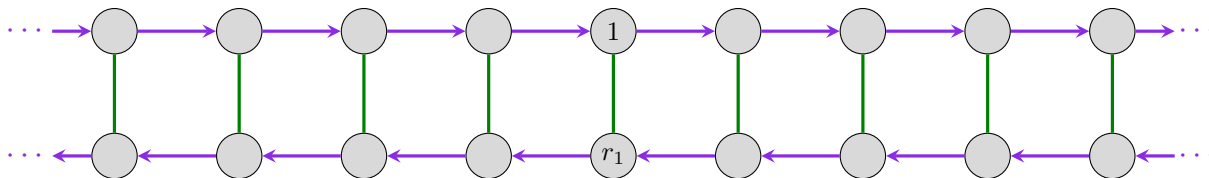
$$\mathbf{Frz}_2 = \langle g, h \mid h^2 = 1, gh = hg^{-1} \rangle = \langle g, r \mid r^2 = 1, gr = rg^{-1} \rangle,$$

where $h = h_0$ and $r = r_1$.

- (a) Label the following Cayley diagram for $\mathbf{Frz}_2 = \langle g, h \rangle$ with elements of the form g^i , h_j , and r_k for $i, j, k \in \mathbb{Z}$.



- (b) Repeat the previous part for the following Cayley diagram:



- (c) Draw a Cayley diagram for $\mathbf{Frz}_2 = \langle g, h, r \rangle$ and label the nodes with actions of the form g^i , h_j , and r_k , for $i, j, k \in \mathbb{Z}$.

- (d) Determine which symmetries $g^i h g^{-i}$ and $g^i r g^{-i}$ are for each $i \in \mathbb{Z}$.