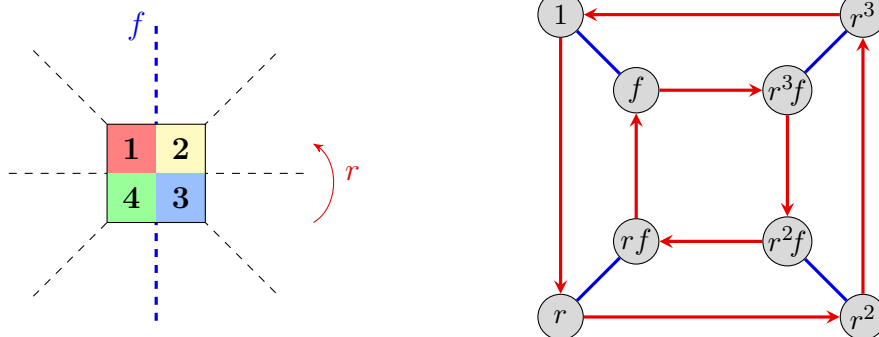
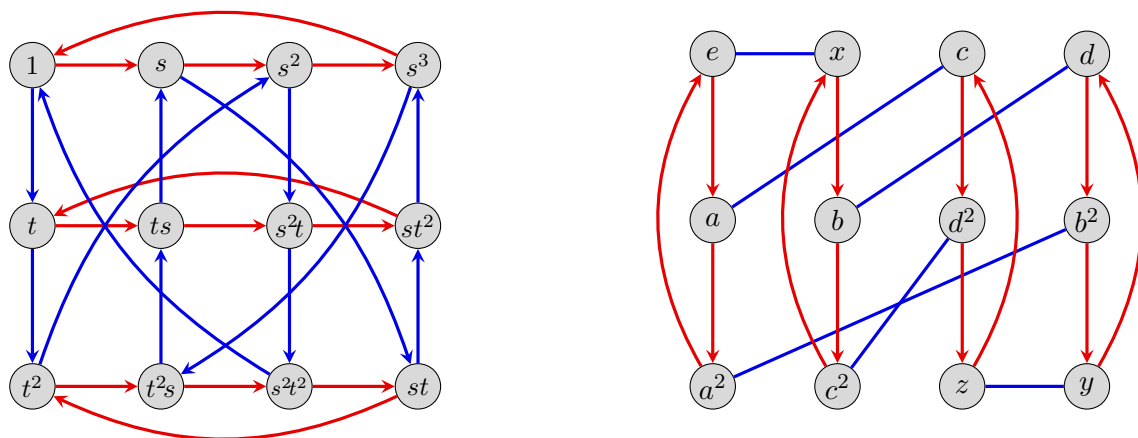


1. The eight symmetries of a square form a group that we will call  $\mathbf{Sq}$ , generated by a  $90^\circ$  counterclockwise rotation  $r$ , and a horizontal flip  $f$ . A Cayley diagram is shown below.

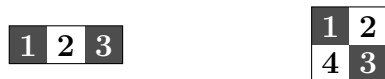


- For each axis of reflection, express the symmetry across it in terms of  $r$  and  $f$ .
  - Find all *minimal* generating sets. [Hint: There are 12.]
  - Let  $s = f$  and  $t = r^3f = fr$ . Draw a Cayley diagram using  $s$  and  $t$  as generators.
  - Write a presentation of the form  $\mathbf{Sq} = \langle r, f \mid \dots \rangle$ .
  - Write a presentation of the form  $\mathbf{Sq} = \langle s, t \mid \dots \rangle$ .
  - Construct a *Cayley table* for this group, ordered  $1, r, r^2, r^3, f, rf, r^2f, r^3f$ . Describe how the rotations and reflections are “clustered” in this table.
2. The Cayley diagrams of two groups of size 12 are shown below.



- Create a Cayley table for each group. Order the elements in the first group by  $1, t^2, s^2t^2, t, s^2, s^2t^2, s, st^2, t^2s, st, s^3, ts$  and those in second by  $e, x, y, z, a, b, c, d, a^2, b^2, c^2, d^2$ .
- Find the inverse of each element.
- Find the *order* of each  $g$ : the minimal  $k > 0$  such that  $g^k = e$ , denoted  $|g|$ .
- Write a presentation for each group.
- Determine whether or not these two groups are isomorphic. Justify your answer.
- Squint your eyes. Do you see any patterns in these tables?

3. In this problem, we will define two variations of the **Coin**<sub>2</sub> group from lecture. We will consider two types of tiles, and declare the following to be the “home state” of each:



Our first group is **Coin**<sub>3</sub> =  $\langle c, t \rangle$ , where  $c$  “cyclically shifts” the entries, and  $t$  “toggles” the color of the leftmost square:

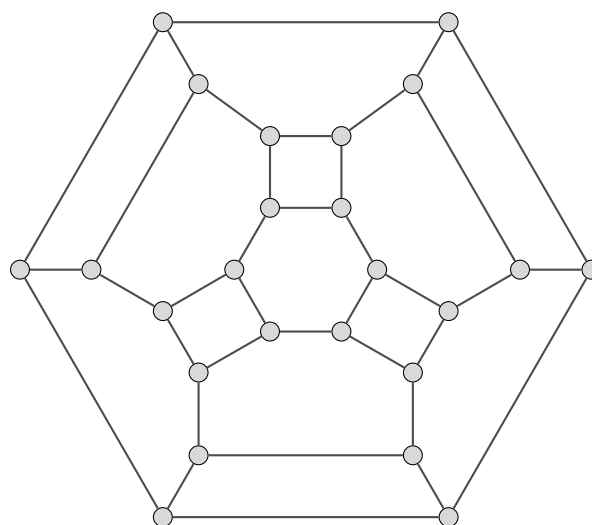
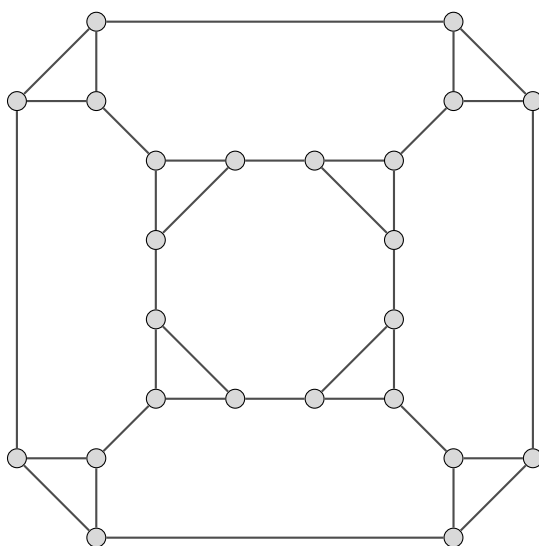


Our second group is **Box**<sub>2</sub> =  $\langle r, s \rangle$ , where  $r$  “rotates” the squares counterclockwise, and  $s$  “swaps” the squares on the top row.



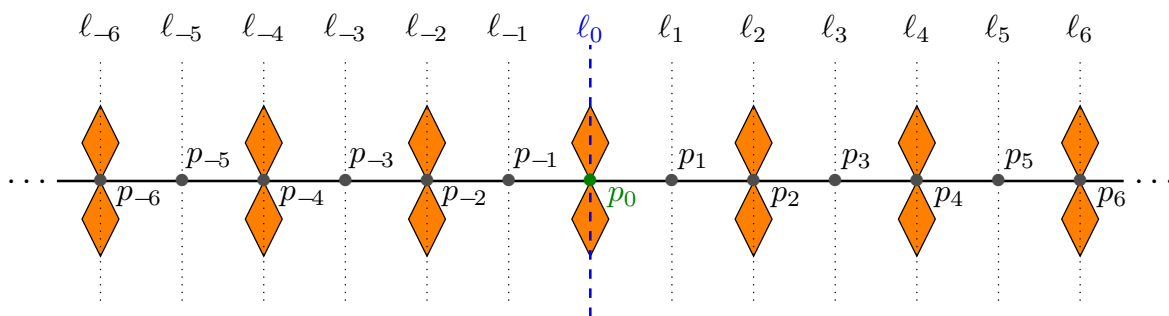
Note that the square tiles don’t actually need to be shaded. An alternate way to denote the colors of the  $3 \times 1$  dominos is to underline any number with a black background. For example, using this convention, the “home state” would be written 1 2 3.

- (a) Both of these groups have 24 actions. Draw a Cayley diagram for each, with the nodes labeled by configurations. It is helpful to know that the one for **Coin**<sub>3</sub> can be arranged on a *truncated cube*, whose skeleton is shown below (left). A Cayley diagram for **Box**<sub>2</sub> can be arranged on a *truncated octahedron*, shown below (right).



- (b) Write down a presentation for each of these groups.
- (c) Are these groups isomorphic? Justify your answer.

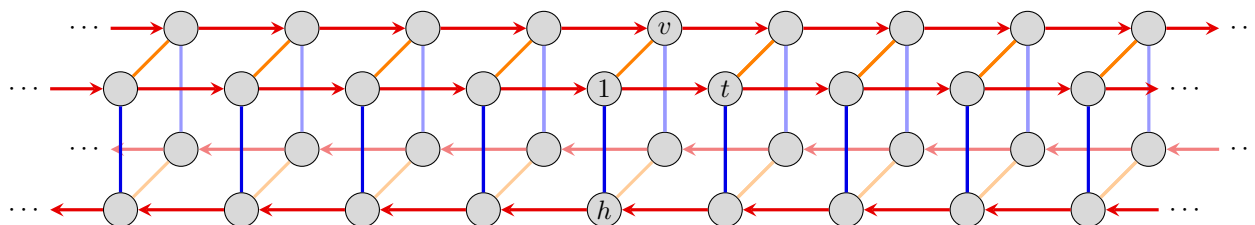
4. Consider the frieze shown below:



Let  $t$  be a minimal translation to the right,  $h_i$  a reflection across  $\ell_i$ , and  $r_j$  a  $180^\circ$  rotation around  $p_j$ . Let  $v$  be the vertical reflection and  $g = tv$  the minimal glide reflection right. A presentation for the frieze group is

$$\mathbf{Frz}_1 := \langle t, h, v \mid v^2 = h^2 = 1, th = ht^{-1}, tv = vt, hv = vh \rangle,$$

where  $h = h_0$ . A Cayley diagram is shown below.

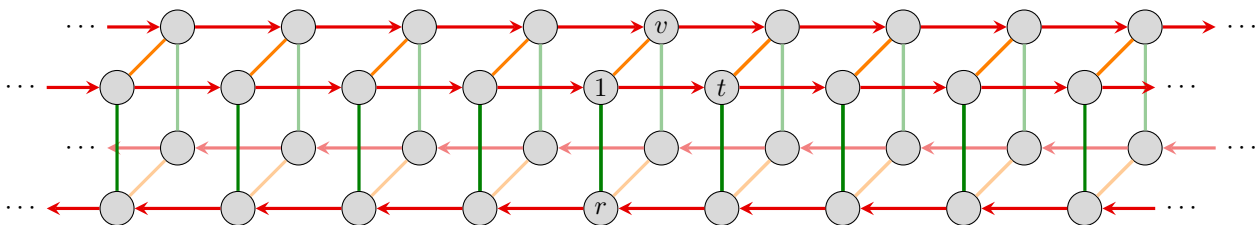


(a) Every symmetry is either a translation  $t^i$ , glide reflection  $g^j$ , rotation  $r_k$ , horizontal reflection  $h_\ell$ , or the vertical reflection  $v$ . Label the vertices of this Cayley diagram with elements written in this form.

(b) Now, repeat the previous part, but with for the Cayley diagram for the presentation

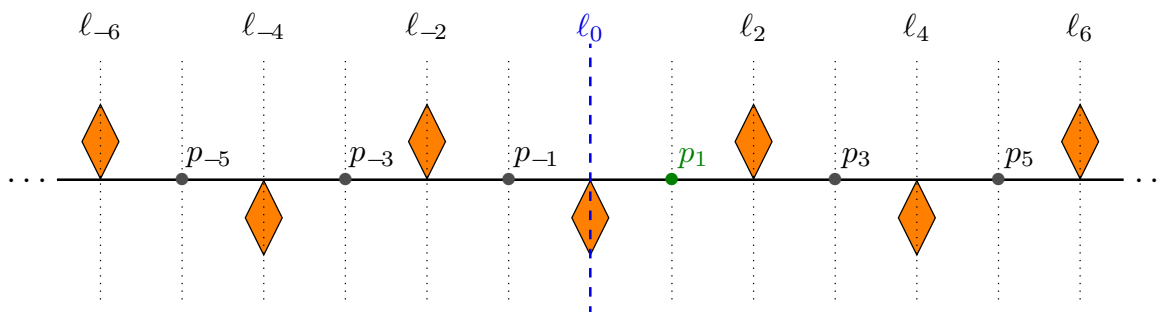
$$\mathbf{Frz}_1 := \langle t, r, v \mid v^2 = r^2 = 1, tr = rt^{-1}, tv = vt, rv = vr \rangle,$$

where  $r = r_0$ . A Cayley diagram is shown below.



(c) Determine which symmetries  $t^i h t^{-i}$  and  $t^i r t^{-i}$  are for each  $i \in \mathbb{Z}$ .

5. The subgroup  $\mathbf{Frz}_2 = \langle g, h \rangle$  of the frieze group from the previous problem is the symmetry group of the following frieze:

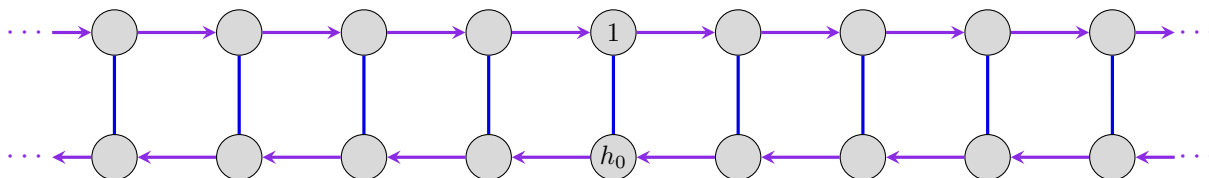


Two presentations for this frieze group are

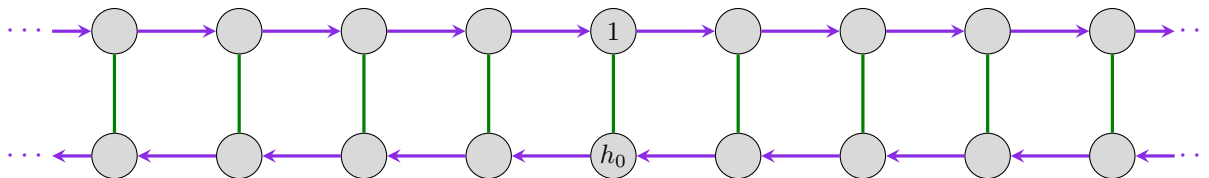
$$\mathbf{Frz}_2 = \langle g, h \mid h^2 = 1, gh = hg^{-1} \rangle = \langle g, r \mid r^2 = 1, gr = rg^{-1} \rangle,$$

where  $h = h_0$  and  $r = r_1$ .

- (a) Label the following Cayley diagram for  $\mathbf{Frz}_2 = \langle g, h \rangle$  with elements of the form  $g^i$ ,  $h_j$ , and  $r_k$  for  $i, j, k \in \mathbb{Z}$ .



- (b) Repeat the previous part for the following Cayley diagram:



- (c) Draw a Cayley diagram for  $\mathbf{Frz}_2 = \langle g, h, r \rangle$  and label the nodes with actions of the form  $g^i$ ,  $h_j$ , and  $r_k$ , for  $i, j, k \in \mathbb{Z}$ .
- (d) Determine which symmetries  $g^i h g^{-i}$  and  $g^i r g^{-i}$  are for each  $i \in \mathbb{Z}$ .