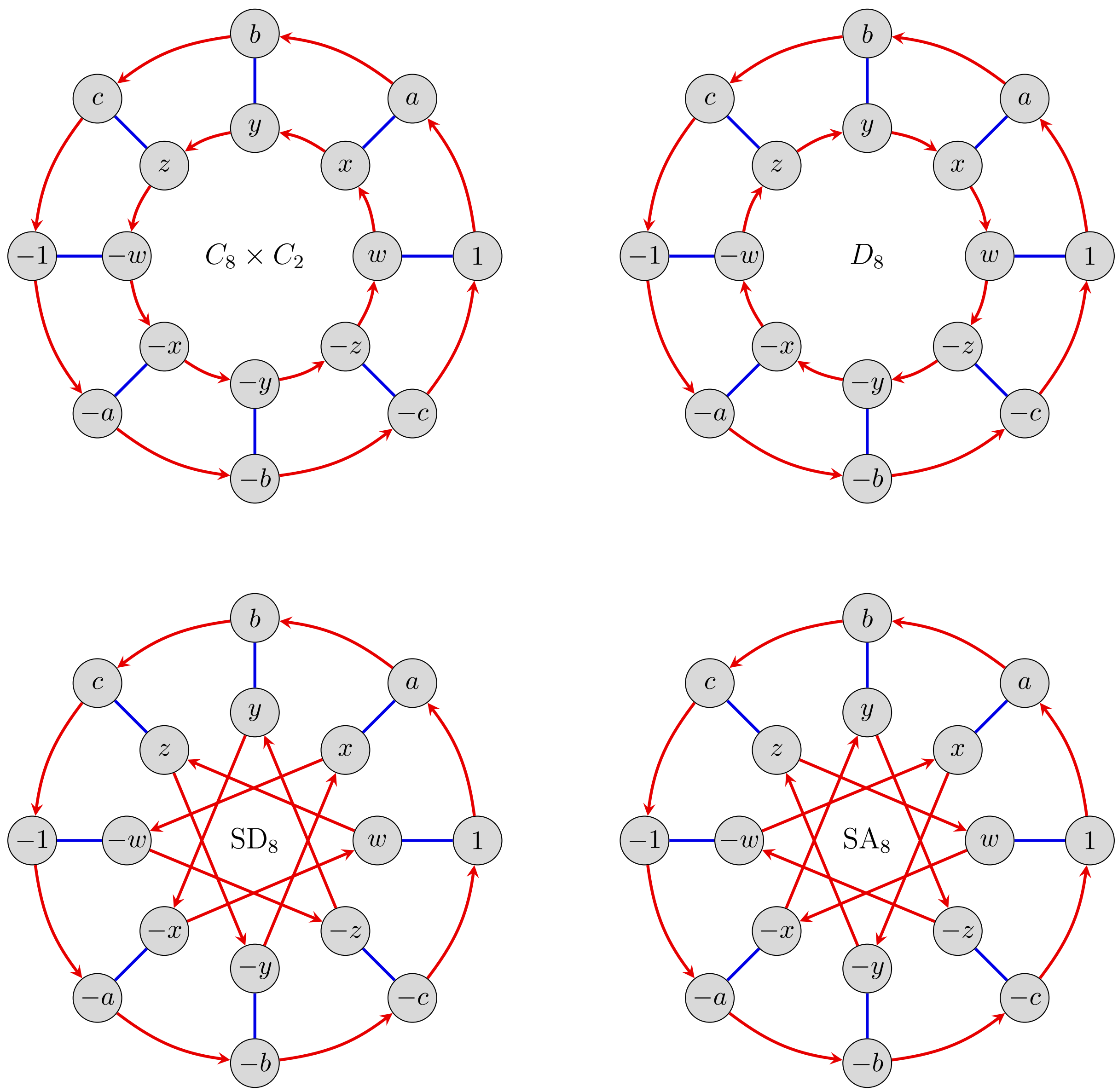
1. Cayley diagrams for the four semidirect products of C_8 with C_2 are shown below, with a different labeling scheme on their nodes.



For all four of these groups, identifying each element with its "negative" yields a "quotient group" of order 8, like what we did with the dicyclic group $\mathrm{Dic}_8 = Q_{16}$ in HW 2. Construct a Cayley table and Cayley diagram for each of these quotient groups, using the elements

$$\pm 1, \quad \pm a, \quad \pm b, \quad \pm c, \quad \pm w, \quad \pm x, \quad \pm y, \quad \pm z,$$

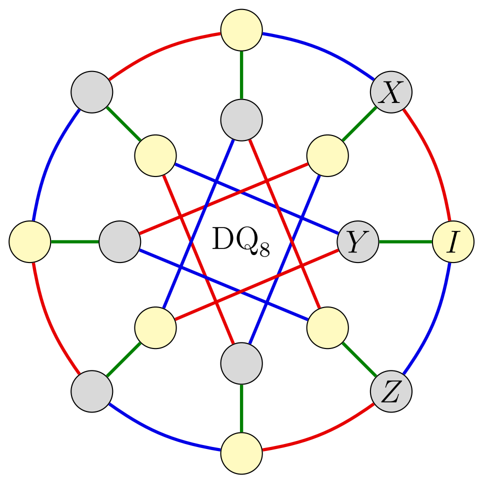
and determine to which familiar group each is isomorphic.

2. The *diquaternion group* DQ_8 can be constructed from the matrices from our standard representation of $Q_8 = \langle R, S, T\rangle$, along with the reflection matrix for f in D_n. That is,

$$\mathrm{DQ}_8 \cong \langle i, j, k, f\rangle \cong \left\langle \underbrace{\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}}_{R=R_4}, \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{S}, \underbrace{\begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}}_{T=RS}, \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{F} \right\rangle.$$

A Cayley diagram for this group, generated by the *Pauli matrices* X, Y, Z, is shown below.

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



The nodes shown in yellow node highlight the subgroup isomorphic to Q_8. For this problem, the use of an online matrix calculator, like `https://matrixcalc.org`, that can handle complex exponential inputs, is strongly recommended.

(a) The elements of DQ_8 can also be written as

$$\big\{ \pm I,\ \pm iI,\ \pm X,\ \pm iX,\ \pm Y,\ \pm iY,\ \pm Z,\ \pm iZ \big\}.$$

Label the remaining nodes in the Cayley diagram. Then, on a new diagram, label the nodes with

$$\big\{ \pm I,\ \pm R,\ \pm S,\ \pm T,\ \pm F,\ \pm RF,\ \pm SF,\ \pm TF \big\}.$$

(b) Now, label the nodes in the Cayley diagram as was done in the previous problem, and carry out the "quotient process."

(c) The dicyclic group Dic_n is only defined when n is even. However, if we try to define

$$\mathrm{Dic}_3 := \left\langle \begin{bmatrix} \zeta_3 & 0 \\ 0 & \bar{\zeta}_3 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\rangle,$$

then the result is still a group. Determine which group this is, with justification.

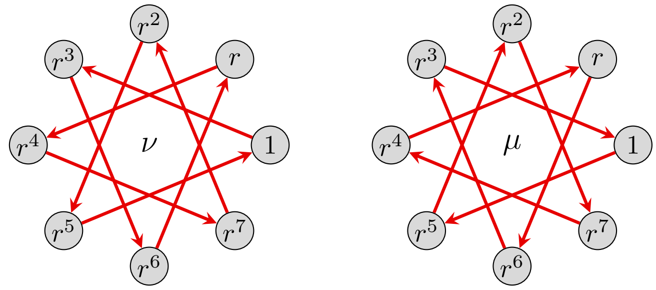
(d) The semidihedral group SD_n is only defined when $n = 2^m$. However, if we define

$$\mathrm{SD}_3 := \left\langle \begin{bmatrix} \zeta_3 & 0 \\ 0 & -\bar{\zeta}_3 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\rangle,$$

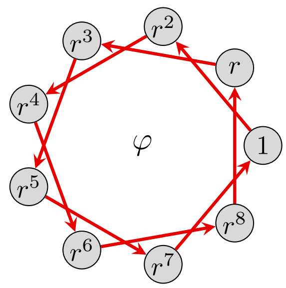
then this is a group of order order 24. Construct a Cayley diagram for this group.

3. The automorphism group of C_n is isomorphic to U_n, the multiplicative group of integers modulo n, from HW 2, #2. For each of the following, construct a Cayley diagram of $\mathrm{Aut}(C_n)$ with the nodes labeled by re-wirings, and a Cayley table for this group.

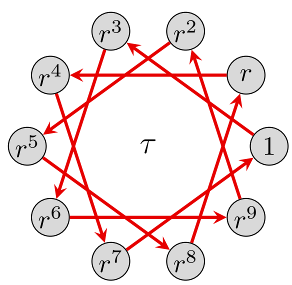
   (a) $\mathrm{Aut}(C_8) = \langle \nu, \mu \rangle$, defined by



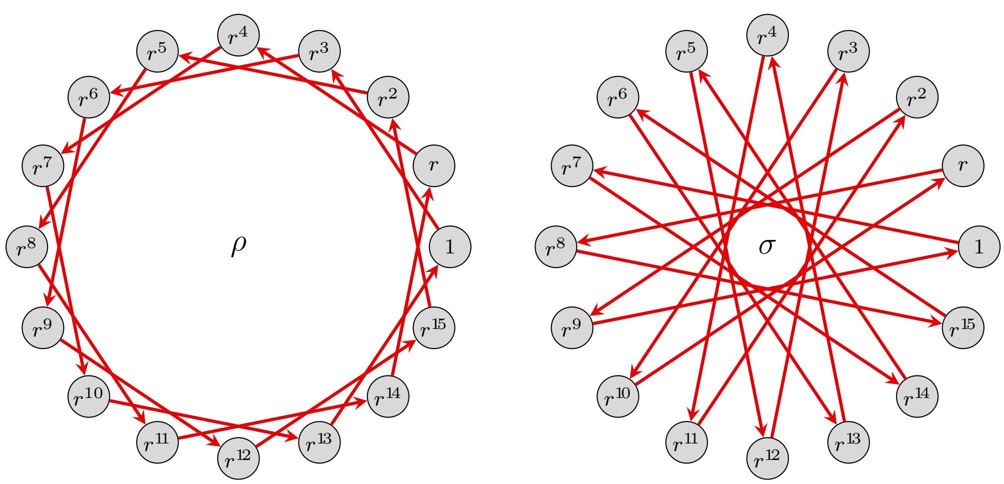
   (b) $\mathrm{Aut}(C_9) = \langle \varphi \rangle$, defined by



   (c) $\mathrm{Aut}(C_{10}) = \langle \tau \rangle$, defined by



   (d) $\mathrm{Aut}(C_{16}) = \langle \rho, \sigma \rangle$, defined by



4. Construct a nonabelian semidirect product of $C_9 = \langle r \rangle$ with $C_3 = \langle s \rangle$. Find all possible labeling maps $\theta \colon C_3 \to \mathrm{Aut}(C_9)$, and specify which one you are using. Include a Cayley diagram of C_3 with the nodes labeled by $\theta(s^j)$, and a Cayley diagram of $C_9 \rtimes_\theta C_3$, with the nodes labeled by $r^i s^j$.