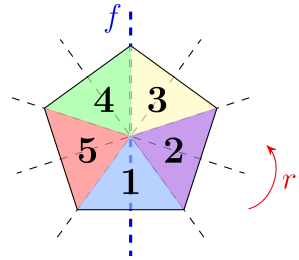
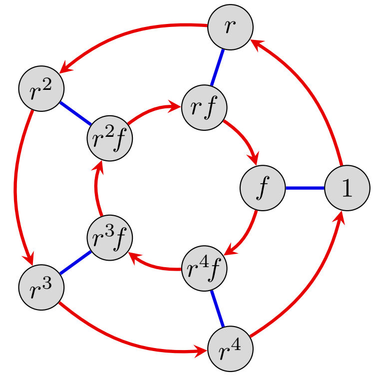
1. All of the subgroups of D_5 should be visually apparent from thinking about symmetries of a regular pentagon, shown below at left. At right is a Cayley diagram.



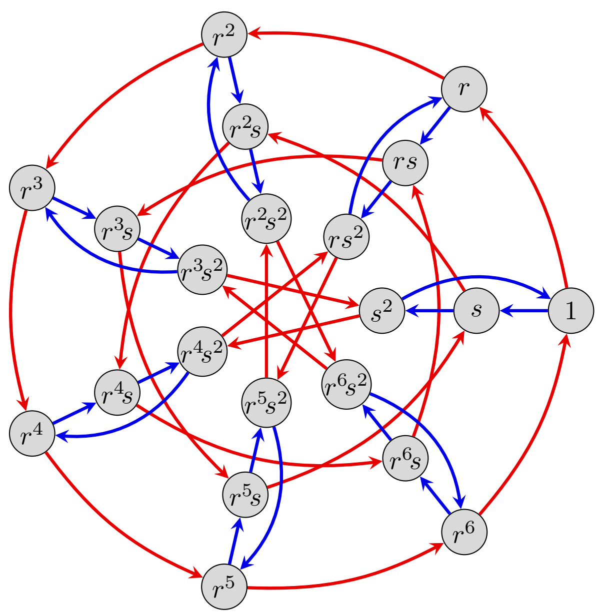


   (a) Construct a subgroup lattice for D_5. Label each edge from H to K with $[H : K]$.

   (b) Find the left and right cosets of the subgroups $\langle f \rangle$ and $\langle r \rangle$, and find their normalizers.

   (c) Two subgroups $H, K \leq G$ are *conjugate* if $K = gHg^{-1} := \{ghg^{-1} \mid h \in H\}$ for some $g \in G$. This defines an equivalence relation on the set of subgroups called *conjugacy classes*. Partition the subgroups of D_5 into conjugacy classes.

2. The Cayley diagram of the smallest non-abelian group, $G = C_7 \rtimes C_3$, is shown below.



   (a) On a blank Cayley diagram, label nodes with the order of the corresponding elements.

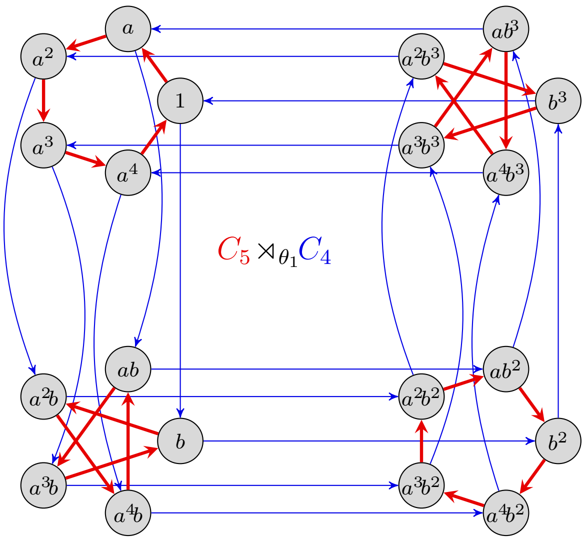
   (b) Construct a subgroup lattice and label each edge with the corresponding index.

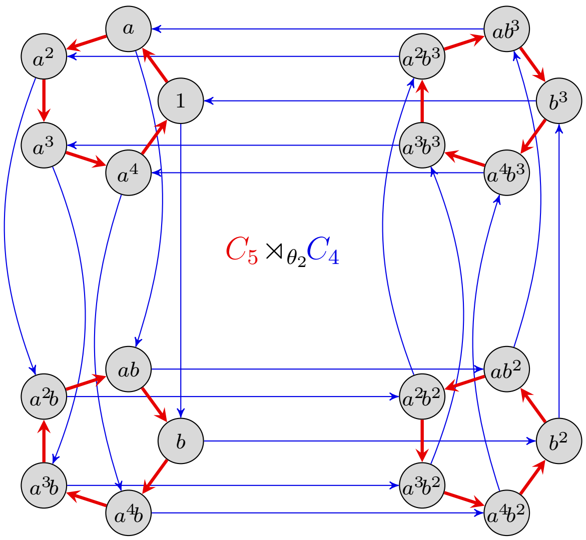
   (c) Find the left and right cosets of the subgroups $\langle s \rangle$ and $\langle r \rangle$, and their normalizers.

   (d) Partition the subgroups into conjugacy classes, and denote this on your lattice.

   (e) Repeat the previous parts for the group $G = C_9 \rtimes C_3$, whose Cayley diagram you constructed on the previous assignment. It is helpful to know that it has four subgroups of order 9 and four subgroups of order 3.

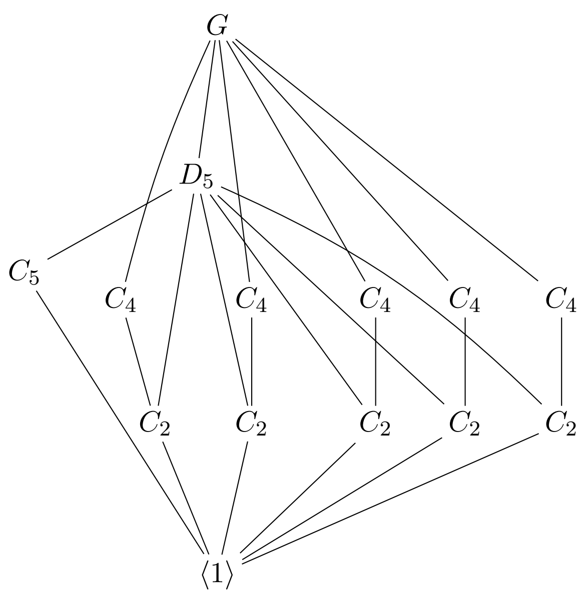
3. Consider two semidirect products of C_5 with C_4, whose Cayley diagrams are shown below.

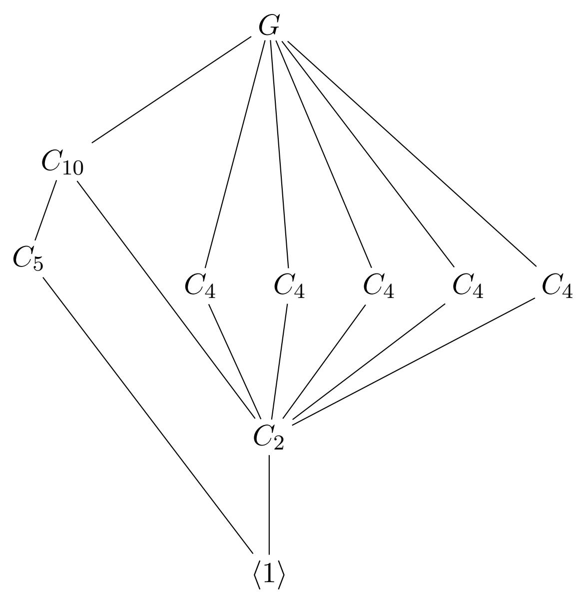




(a) On blank Cayley diagrams, label the order of each element.

(b) The subgroup lattices of these two groups are shown below, not necessarily in the right order. Determine which lattice corresponds to which Cayley diagram (with justification), and then re-draw them with the subgroups written by generators.





(c) Determine which group each of these is isomorphic to, and which elements a and b correspond to. Recall that there are only three nonabelian groups of order 20:

$$D_{10} = \langle r, f \mid r^{10} = f^2 = 1,\ rfr = f \rangle, \qquad \mathrm{Dic}_{10} = \langle r, s \mid r^{10} = s^4 = 1,\ r^5 = s^2 \rangle,$$

$$\mathrm{AGL}_1(\mathbb{Z}_5) = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle \leq \mathrm{GL}_2(\mathbb{Z}_5).$$

Write a presentation for both groups in this problem, in terms of a and b.

4. Prove the following, algebraically (that is, do not refer to Cayley diagrams):

   (a) If $\mathcal{H}$ is a collection of subgroups of G, then $\displaystyle\bigcap_{H_\alpha \in \mathcal{H}} H_\alpha$ is a subgroup of G.

   (b) For any (possibly infinite) subset $S \subseteq G$, the subgroup generated by S is defined as

   $$\langle S \rangle := \big\{ s_1^{e_1} s_2^{e_2} \cdots s_k^{e_k} \mid s_i \in S,\ e_i \in \{-1, 1\} \big\}.$$

   That is, $\langle S \rangle$ consists of all finite "words" that can be written using the elements in S and their inverses. Note that the s_i's need not be distinct. Prove that

   $$\langle S \rangle = \bigcap_{S \subseteq H_\alpha \leq G} H_\alpha,$$

   where the intersection is taken over all subgroups of G that contain S. [*Hint*: To prove that $A = B$, you need to show that that $A \subseteq B$ and $B \subseteq A$.]