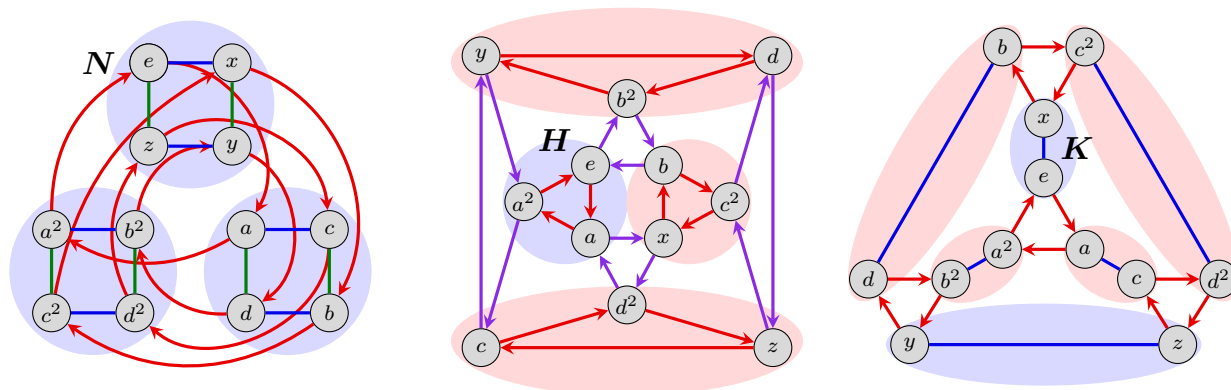


1. Below are three Cayley diagrams of A_4 , each highlighting the left cosets of a different subgroup. We can take $a = (123)$, $b = (234)$, $x = (12)(34)$, and $z = (13)(24)$.



- (a) For each subgroup shown above, partition A_4 into right cosets. Write them as subsets of A_4 , consisting of permutations in cycle notation. Also, highlight them by colors on a fresh copy of the Cayley diagrams.
- (b) For each left coset xH , illustrate the construction of the conjugate subgroup xHx^{-1} on a fresh copy of the Cayley diagram.

2. Show that $A \times \{1_B\}$ is a normal subgroup of $A \times B$, where 1_B is the identity element of B . That is, show first that it is a subgroup, and then that it is normal.

3. Let G be a group, not necessarily finite, and $H \leq G$.

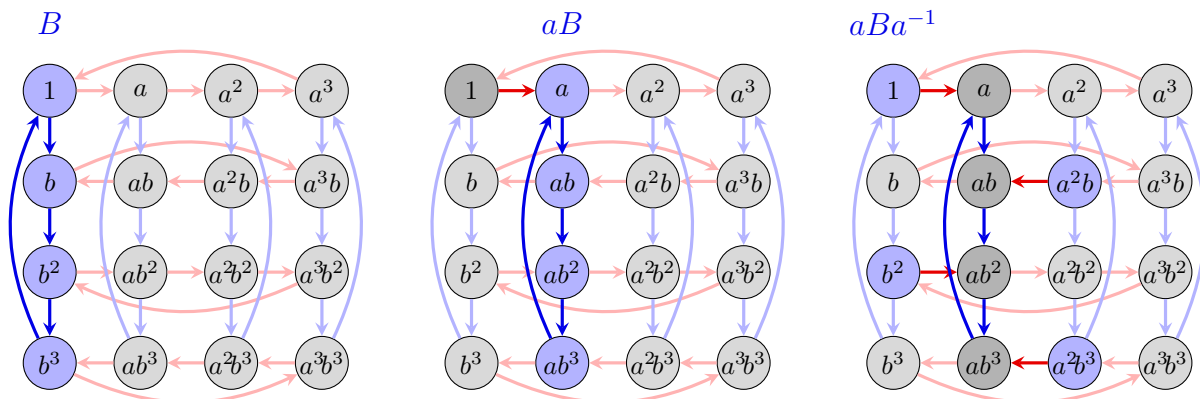
- (a) Show that for any fixed $x \in G$, we have an equality $\{gx \mid g \in G\} = G$ of sets.
- (b) Show that the subgroup

$$N := \bigcap_{x \in G} xHx^{-1}$$

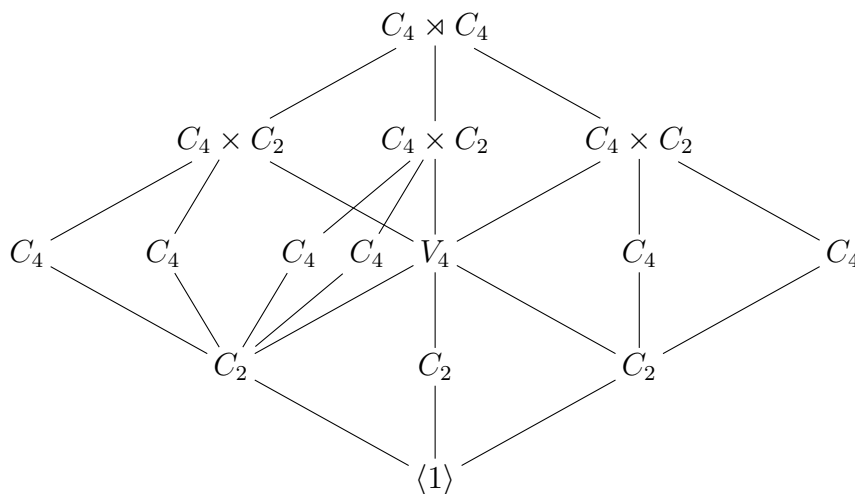
is normal in G .

- (c) Show that every normal subgroup $K \trianglelefteq G$ contained in H is contained in N . In other words, N is the largest normal subgroup of G contained in H .

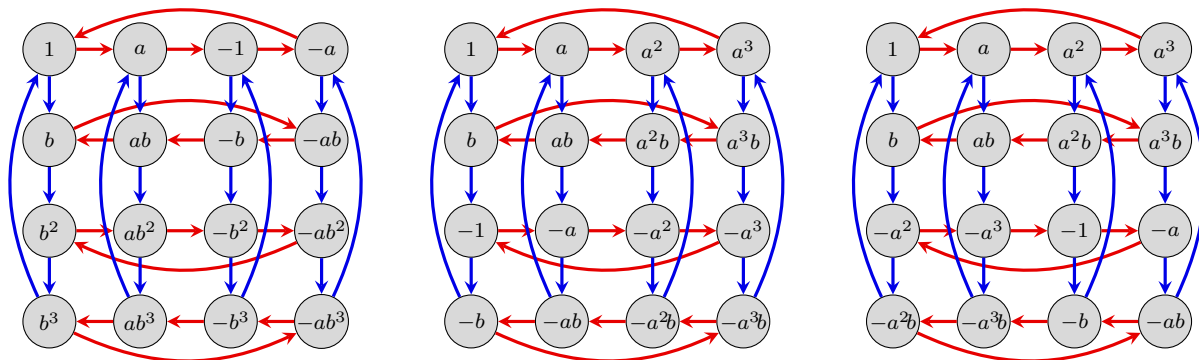
4. Shown below is a Cayley diagram for $G = C_4 \rtimes C_4 = \langle a, b \rangle$, and the construction of the conjugate subgroup aBa^{-1} , where $B = \langle b \rangle$.



- (a) Find all conjugates of the subgroups $\langle ab \rangle$ and $\langle a^2, b^2 \rangle$, and illustrate the construction of each on the Cayley diagram, as was done above for $B = \langle b \rangle$.
- (b) The subgroup lattice of $G = \langle a, b \rangle$ is shown below. Re-draw this with the subgroups written by generator(s). Then partition the subgroups into conjugacy classes, and fully justify your answer.



- (c) Without computing left or right cosets, i.e., only from the knowledge of conjugacy classes, find the normalizer of each subgroup. Justify your answer.
- (d) Find the center $Z(G)$ of this group.
- (e) Shown below are three copies of the Cayley diagram for G , each with a peculiar labeling of the nodes. In each diagram, pair of nodes that differ by a sign represent a left coset of one of the three order-2 subgroups, all of which are normal.



For each diagram, construct a multiplication table and Cayley diagram consisting of these eight “cosets”. Determine what the resulting *quotient group* is isomorphic to, and describe where that subgroup lattice appears, hiding in the lattice for G .

5. Let H be a subgroup of G . Given two fixed elements $a, b \in G$, define the sets

$$aHbH = \{ah_1bh_2 \mid h_1, h_2 \in H\} \quad \text{and} \quad abH = \{abh \mid h \in H\}.$$

Show that if $H \trianglelefteq G$, then $aHbH = abH$.