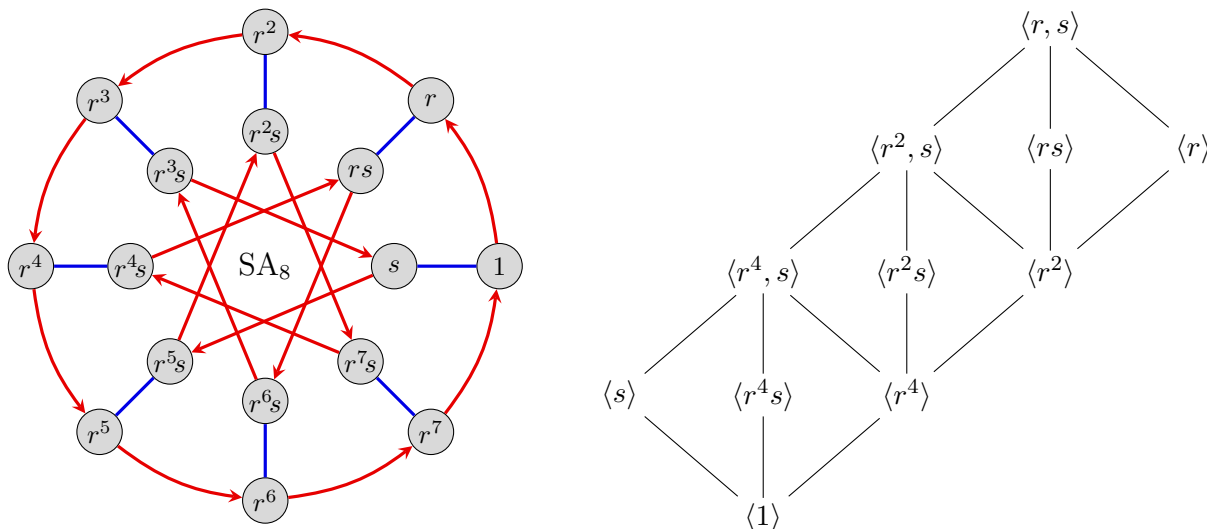


1. Let G be the *semiabelian group* of order 16, defined by the presentation

$$SA_8 = \langle r, s \mid r^8 = s^2 = 1, srs = r^5 \rangle,$$

A Cayley diagram and subgroup lattice are shown below.



- (a) The subgroups $V = \langle r^4, s \rangle$, $H = \langle r^2 s \rangle$, $K = \langle r^2 \rangle$, and $N = \langle r^4 \rangle$ are all normal. For the first three, highlight the cosets on a fresh Cayley diagram by colors.
- (b) Construct a Cayley table for the quotient of G by each of these subgroups. Then draw a Cayley diagram for each, labeling the nodes with elements (i.e., cosets).
- (c) Let $N = \langle r^4 \rangle$. The shaded region below shows an order-4 cyclic subgroup of G/N , generated by the element rN , and how the union of these four cosets is the order-8 subgroup $\langle r \rangle$ of G . Construct analogous tables for the other five non-trivial proper subgroups of G/N , and then draw the subgroup lattice of G/N .

$r^3 N$	$r^3 s N$
$r^2 N$	$r^2 s N$
$r N$	$r s N$
N	$s N$

$\langle rN \rangle \leq G/N$

r^3	r^7	$r^3 s$	$r^7 s$
r^2	r^6	$r^2 s$	$r^6 s$
r	r^5	$r s$	$r^5 s$
1	r^4	s	$r^4 s$

$\langle r \rangle / N \leq G/N$

r^3	r^7	$r^3 s$	$r^7 s$
r^2	r^6	$r^2 s$	$r^6 s$
r	r^5	$r s$	$r^5 s$
1	r^4	s	$r^4 s$

$\langle r \rangle \leq G$

- (d) For each subgroup H/N from Part (c), determine what the quotient of G/N by H/N is isomorphic to? Justify your answer.
- (e) One step of Part (c) consisting of starting with G , taking the quotient by N , and then taking the subgroup generated by $r^2 N$ and $s N$. Compare and contrast this to doing these steps in the reverse order. That is: *start with G , first take the subgroup $\langle r^2, s \rangle$, and then take the quotient by N .*
- (f) Repeat Part (c) for the subgroups $\langle r^4, s \rangle$, $\langle r^2 s \rangle$, and $\langle r^2 \rangle$ of G .

2. Show that there is no embedding $\phi: \mathbb{Z}_n \hookrightarrow \mathbb{Z}$.
3. All of the following can be done by defining an explicit map, showing that it is a homomorphism, and a bijection.
- (a) Show that $A \times B \cong B \times A$.
 - (b) Show that $xHx^{-1} \cong H$, for any $H \leq G$. Conclude that $|xy| = |yx|$ for any $x, y \in G$.
 - (c) Show that $(\mathbb{Q}^*, \cdot) \cong (\mathbb{Q}^+, \cdot) \times C_2$. Recall that \mathbb{Q}^* and \mathbb{Q}^+ are the nonzero and positive rational numbers, respectively, and $C_2 = \{1, -1\}$.
4. Let $\phi: G \rightarrow H$ be a homomorphism, and $N \trianglelefteq H$.
- (i) Show that the set $\phi^{-1}(N) := \{g \in G \mid \phi(g) \in N\}$ is a subgroup of G .
 - (ii) Show that $\phi^{-1}(N)$ is a *normal* subgroup of G .
 - (iii) Show by example that if $M \trianglelefteq G$, then $\phi(M)$ need not be a normal subgroup of H .
5. In this exercise, you will show that if A and B are normal subgroups and $AB = G$, then

$$G/(A \cap B) \cong (G/A) \times (G/B).$$

- (a) Consider the following map:

$$\phi: AB \longrightarrow (G/A) \times (G/B), \quad \phi(g) = (gA, gB).$$

Show that ϕ is a homomorphism.

- (b) Show that ϕ is surjective. That is, given any (g_1A, g_2B) , show that there is some $g = ab \in AB$ such that $\phi(g) = (g_1A, g_2B)$. [*Hint*: Try $g = a_2b_1$.]
- (c) Find $\text{Ker}(\phi)$ [you need to verify your answer is correct] and then apply the fundamental homomorphism theorem.