Lecture 4.2: Kernels

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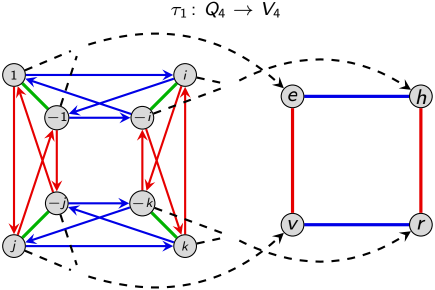
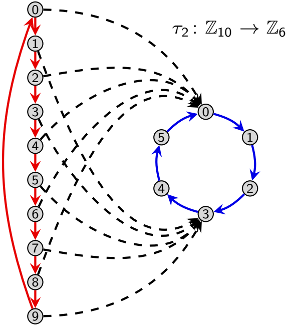
Math 4120, Modern Algebra

Quotient maps

Consider a homomorphism where more than one element of the domain maps to the same element of codomain (i.e., non-embeddings).

Here are some examples.





Non-embedding homomorphisms are called quotient maps (as we'll see, they correspond to our quotient process).

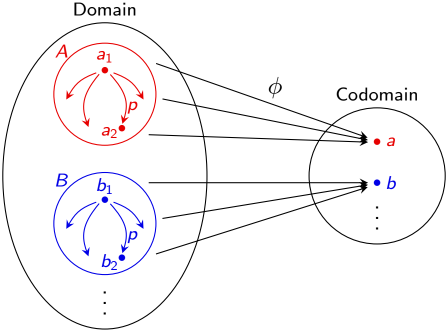
Preimages

Definition

If $\phi\colon G \to H$ is a homomorphism and $h \in \mathrm{Im}(\phi) < H$, define the preimage of h to be the set

$$\phi^{-1}(h) := \{g \in G : \phi(g) = h\}\,.$$

Observe in the previous examples that the preimages all had the same structure. This always happens.



The preimage of $1_H \in H$ is called the kernel of ϕ, denoted $\mathrm{Ker}\,\phi$.

Preimages

Observation 1

All preimages of ϕ have the same structure.

Proof (sketch)

Pick two elements $a, b \in \phi(G)$, and let $A = \phi^{-1}(a)$ and $B = \phi^{-1}(b)$ be their preimages.

Consider any path $a_1 \xrightarrow{p} a_2$ between elements in A. For any $b_1 \in B$, there is a corresponding path $b_1 \xrightarrow{p} b_2$. *We need to show that $b_2 \in B$.*

Since homomorphisms preserve structure, $\phi(a_1) \xrightarrow{\phi(p)} \phi(a_2)$. Since $\phi(a_1) = \phi(a_2)$, $\phi(p)$ is the *empty path*.

Therefore, $\phi(b_1) \xrightarrow{\phi(p)} \phi(b_2)$, i.e., $\phi(b_1) = \phi(b_2)$, and so by definition, $b_2 \in B$. $\square$

Clearly, G is partitioned by preimages of ϕ. Additionally, we just showed that they all have the same structure. (Sound familiar?)

Preimages and kernels

Definition

The kernel of a homomorphism $\phi \colon G \to H$ is the set

$$\mathrm{Ker}(\phi) := \phi^{-1}(e) = \{k \in G : \phi(k) = e\}\,.$$

Observation 2

(i) The preimage of the identity (i.e., $K = \mathrm{Ker}(\phi)$) is a subgroup of G.

(ii) All other preimages are left cosets of K.

Proof (of (i))

Let $K = \mathrm{Ker}(\phi)$, and take $a, b \in K$. We must show that K satisfies 3 properties:

Identity: $\phi(e) = e$. $\checkmark$

Closure: $\phi(ab) = \phi(a)\,\phi(b) = e \cdot e = e$. $\checkmark$

Inverses: $\phi(a^{-1}) = \phi(a)^{-1} = e^{-1} = e$. $\checkmark$

Thus, K is a subgroup of G. $\square$

Kernels

Observation 3

Ker(ϕ) is a normal subgroup of G.

Proof

Let $K = \text{Ker}(\phi)$. We will show that if $k \in K$, then $gkg^{-1} \in K$. Take any $g \in G$, and observe that

$$\phi(gkg^{-1}) = \phi(g)\,\phi(k)\,\phi(g^{-1}) = \phi(g) \cdot e \cdot \phi(g^{-1}) = \phi(g)\phi(g)^{-1} = e\,.$$

Therefore, $gkg^{-1} \in \text{Ker}(\phi)$, so $K \lhd G$. $\square$

Key observation

Given any homomorphism $\phi \colon G \to H$, we can *always* form the quotient group $G/\,\text{Ker}(\phi)$.

Quotients: via multiplication tables

Recall that $C_2 = \{e^{0\pi i}, e^{1\pi i}\} = \{1, -1\}$. Consider the following (quotient) homomorphism:

$$\phi \colon D_4 \longrightarrow C_2\,, \qquad \text{defined by } \phi(r) = 1 \text{ and } \phi(f) = -1\,.$$

Note that $\phi(\text{rotation}) = 1$ and $\phi(\text{reflection}) = -1$.

The quotient process of "shrinking D_4 down to C_2" can be clearly seen from the multiplication tables.

	e	r	r^2	r^3	f	rf	r^2f	r^3f
e	e	r	r^2	r^3	f	rf	r^2f	r^3f
r	r	r^2	r^3	e	rf	r^2f	r^3f	f
r^2	r^2	r^3	e	r	r^2f	r^3f	f	rf
r^3	r^3	e	r	r^2	r^3f	f	rf	r^2f
f	f	r^3f	r^2f	rf	e	r^3	r^2	r
rf	rf	f	r^3f	r^2f	r	e	r^3	r^2
r^2f	r^2f	rf	f	r^3f	r^2	r	e	r^3
r^3f	r^3f	r^2f	rf	f	r^3	r^2	r	e

	e	r	r^2	r^3	f	rf	r^2f	r^3f
e	e	r	r^2	r^3	f	rf	r^2f	r^3f
r	r	r^2	r^3	e	rf	r^2f	r^3f	f
r^2	r^2	r^3	e	r	r^2f	r^3f	f	rf
r^3	r^3	e	r	r^2	r^3f	f	rf	r^2f
f	f	r^3f	r^2f	rf	e	r^3	r^2	r
rf	rf	f	r^3f	r^2f	r	e	r^3	r^2
r^2f	r^2f	rf	f	r^3f	r^2	r	e	r^3
r^3f	r^3f	r^2f	rf	f	r^3	r^2	r	e

non-flip flip

flip non-flip

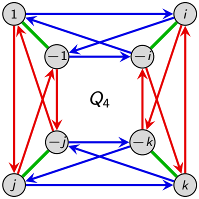
	1	-1
1	1	-1
-1	-1	1

Quotients: via Cayley diagrams

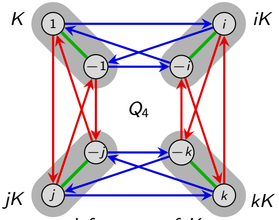
Define the homomorphism $\phi : Q_4 \to V_4$ via $\phi(i) = v$ and $\phi(j) = h$. Since $Q_4 = \langle i, j \rangle$, we can determine where ϕ sends the remaining elements:

$$\begin{aligned} \phi(1) &= e\,, & \phi(-1) &= \phi(i^2) = \phi(i)^2 = v^2 = e\,, \\ \phi(k) &= \phi(ij) = \phi(i)\phi(j) = vh = r\,, & \phi(-k) &= \phi(ji) = \phi(j)\phi(i) = hv = r\,, \\ \phi(-i) &= \phi(-1)\phi(i) = ev = v\,, & \phi(-j) &= \phi(-1)\phi(j) = eh = h\,. \end{aligned}$$

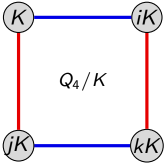
Note that $\operatorname{Ker}\phi = \{-1, 1\}$. Let's see what happens when we quotient out by $\operatorname{Ker}\phi$:



Q_4 organized by the subgroup $K = \langle -1 \rangle$



left cosets of K are near each other



collapse cosets into single nodes

Do you notice any relationship between $Q_4/\operatorname{Ker}(\phi)$ and $\operatorname{Im}(\phi)$?