Chapter 7: Rings

Matthew Macauley

Department of Mathematical Sciences
Clemson University
http://www.math.clemson.edu/~macaule/

Math 4120, Modern Algebra

What is a ring?

A group is a set with a binary operation, satisfying a few basic properties.

Many algebraic structures (numbers, matrices, functions) have two binary operations.

Definition

A ring is an additive (abelian) group R with an additional binary operation (multiplication), satisfying the distributive law:

$$x(y+z) = xy + xz \qquad \text{and} \qquad (y+z)x = yx + zx \quad \forall x, y, z \in R\,.$$

Remarks

- There need not be multiplicative inverses.
- Multiplication need not be commutative (it may happen that $xy \neq yx$).

A few more definitions

If $xy = yx$ for all $x, y \in R$, then R is commutative.

If R has a multiplicative identity $1 = 1_R \neq 0$, we say that "R has identity" or "unity", or "R is a ring with 1."

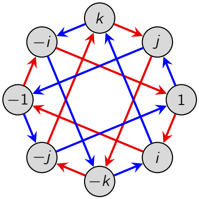
A subring of R is a subset $S \subseteq R$ that is also a ring.

What is a ring?

Examples

1. $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$ are all commutative rings with 1.
2. $\mathbb{Z}_n$ is a commutative ring with 1.
3. For any ring R with 1, the set $M_n(R)$ of $n \times n$ matrices over R is a ring. It has identity $1_{M_n(R)} = I_n$ iff R has 1.
4. For any ring R, the set of functions $F = \{f \colon R \to R\}$ is a ring by defining
$$(f+g)(r) = f(r) + g(r), \qquad (fg)(r) = f(r)g(r)\,.$$
5. The set $S = 2\mathbb{Z}$ is a subring of $\mathbb{Z}$ but it does *not* have 1.
6. $S = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \in \mathbb{R} \right\}$ is a subring of $R = M_2(\mathbb{R})$. However, note that
$$1_R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \text{but} \qquad 1_S = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$
7. If R is a ring and x a variable, then the set
$$R[x] = \left\{ a_n x^n + \cdots + a_1 x + a_0 \mid a_i \in R \right\}$$
is called the polynomial ring over R.

Another example: the Hamiltonians



Recall the (unit) quaternion group:

$$Q_8 = \langle i, j, k \mid i^2 = j^2 = k^2 = -1,\ ij = k \rangle.$$

Allowing addition makes them into a ring $\mathbb{H}$, called the quaternions, or Hamiltonians:

$$\mathbb{H} = \big\{ a + bi + cj + dk \mid a, b, c, d \in \mathbb{R} \big\}.$$

The set $\mathbb{H}$ is isomorphic to a subring of $M_4(\mathbb{R})$, the real-valued 4×4 matrices:

$$\mathbb{H} \cong \left\{ \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\} \subseteq M_4(\mathbb{R}) \,.$$

Formally, we have an embedding $\phi \colon \mathbb{H} \hookrightarrow M_4(\mathbb{R})$ where

$$\phi(i) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \phi(j) = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad \phi(k) = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Just like with groups, we say that $\mathbb{H}$ is represented by a set of matrices.

Units and zero divisors

Informally, a ring is a set where we can add, substract, multiply, but not necessarily divide.

Definition

Let R be a ring with 1. A unit is any $x \in R$ that has a multiplicative inverse. Let $U(R)$ be the set (a multiplicative group) of units of R.

An element $x \in R$ is a left zero divisor if $xy = 0$ for some $y \neq 0$. (Right zero divisors are defined analogously.)

Examples

1. Let $R = \mathbb{Z}$. The units are $U(R) = \{-1, 1\}$. There are no (nonzero) zero divisors.
2. Let $R = \mathbb{Z}_{10}$. Then 7 is a unit (and $7^{-1} = 3$) because $7 \cdot 3 = 1$. However, 2 is not a unit.
3. Let $R = \mathbb{Z}_n$. A nonzero $k \in \mathbb{Z}_n$ is a unit if $\gcd(n, k) = 1$, and a zero divisor otherwise.
4. The ring $R = M_2(\mathbb{R})$ has zero divisors, such as:

$$\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The groups of units of $M_2(\mathbb{R})$ are the invertible matrices.

Group rings

A rich family of examples of rings can be constructed from multiplicative groups.

Let G be a finite (multiplicative) group, and R a commutative ring (usually, $\mathbb{Z}$, $\mathbb{R}$, or $\mathbb{C}$).

The group ring RG is the set of formal linear combinations of groups elements with coefficients from R. That is,

$$RG := \{a_1 g_1 + \cdots + a_n g_n \mid a_i \in R,\ g_i \in G\},$$

where multiplication is defined in the "obvious" way.

For example, let $R = \mathbb{Z}$ and $G = D_4$, and consider $x = r + r^2 - 3f$ and $y = -5r^2 + rf$ in $\mathbb{Z}D_4$.

Their sum is

$$x + y = r - 4r^2 - 3f + rf,$$

and their product is

$$\begin{aligned} xy &= (r + r^2 - 3f)(-5r^2 + rf) = r(-5r^2 + rf) + r^2(-5r^2 + rf) - 3f(-5r^2 + rf) \\ &= -5r^3 + r^2 f - 5r^4 + r^3 f + 15fr^2 - 3frf = -5 - 8r^3 + 16r^2 f + r^3 f. \end{aligned}$$

Group rings

For another example, consider the group ring $\mathbb{R}Q_8$. Elements are formal sums

$$a + bi + cj + dk + e(-1) + f(-i) + g(-j) + h(-k), \qquad a, \ldots, h \in \mathbb{R}.$$

Every choice of coefficients gives a different element in $\mathbb{R}Q_8$.

For example, if all coefficients are zero except $a = e = 1$, we get

$$1 + (-1) \neq 0 \in \mathbb{R}Q_8.$$

In contrast, in the Hamiltonians, $\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$, and so

$$1 + (-1) = [1 + 0i + 0j + 0k] + [(-1) + 0i + 0j + 0k] = (1 - 1) + 0i + 0j + 0k = 0.$$

Therefore, $\mathbb{H}$ and $\mathbb{R}Q_8$ are different rings.

Remark

If $g \in G$ has finite order $|g| = k > 1$, then RG always has zero divisors:

$$(1 - g)(1 + g + \cdots + g^{k-1}) = 1 - g^k = 1 - 1 = 0.$$

RG contains a subring isomorphic to R, and the group of units $U(RG)$ contains a subgroup isomorphic to G.

Types of rings

Definition

A field is a commutiative ring where all nonzero elements have a multiplicative inverse.

If we drop "commutative", the result is called a skew field, or division ring.

An integral domain is a commutative ring with 1 and with no (nonzero) zero divisors. (Think: "field without inverses".)

A field is just a commutative division ring. Moreover:

$$\text{fields} \subsetneq \text{division rings}, \qquad \text{fields} \subsetneq \text{integral domains}.$$

Examples

- Rings that are not integral domains: $\mathbb{Z}_n$ (composite n), $2\mathbb{Z}$, $M_n(\mathbb{R})$, $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{H}$.
- Integral domains that are not fields (or even division rings): $\mathbb{Z}$, $\mathbb{Z}[x]$, $\mathbb{R}[x]$, $\mathbb{R}[[x]]$ (formal power series).
- Division ring but not a field: $\mathbb{H}$.

Cancellation

When doing basic algebra, we often take for granted basic properties such as cancellation:

$$ax = ay \quad \Longrightarrow \quad x = y.$$

However, *this need not hold in all rings!*

Examples where cancellation fails

- In $\mathbb{Z}_6$, note that $2 = 2 \cdot 1 = 2 \cdot 4$, but $1 \neq 4$.
- In $M_2(\mathbb{R})$, note that $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$.

However, everything works fine as long as there aren't any (nonzero) zero divisors.

Proposition

Let R be an integral domain and $a \neq 0$. If $ax = ay$ for some $x, y \in R$, then $x = y$.

Proof

If $ax = ay$, then $ax - ay = a(x - y) = 0$.

Since $a \neq 0$ and R has no (nonzero) zero divisors, then $x - y = 0$. $\square$

Finite integral domains

Lemma (HW)

If R is an integral domain and $0 \neq a \in R$ and $k \in \mathbb{N}$, then $a^k \neq 0$. $\square$

Theorem

Every finite integral domain is a field.

Proof

Suppose R is a finite integral domain and $0 \neq a \in R$. It suffices to show that a has a multiplicative inverse.

Consider the infinite sequence $a, a^2, a^3, a^4, \dots,$ which must repeat.

Find $i > j$ with $a^i = a^j$, which means that

$$0 = a^i - a^j = a^j(a^{i-j} - 1).$$

Since R is an integral domain and $a^j \neq 0$, then $a^{i-j} = 1$.

Thus, $a \cdot a^{i-j-1} = 1$. $\square$

Ideals

In group theory, we can quotient out by a subgroup if and only if it is normal.

The analogue of this for rings are (two-sided) ideals.

Definition

A subring $I \subseteq R$ is a left ideal if

$$rx \in I \qquad \text{for all } r \in R \text{ and } x \in I.$$

Right ideals, and two-sided ideals are defined similarly.

If R is commutative, then all left (or right) ideals are two-sided.

We use the term ideal and two-sided ideal synonymously, and write $I \trianglelefteq R$.

Examples

- $n\mathbb{Z} \trianglelefteq \mathbb{Z}$.
- If $R = M_2(\mathbb{R})$, then $I = \left\{ \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} : a, c \in \mathbb{R} \right\}$ is a left, but *not* a right ideal of R.
- The set $\mathrm{Sym}_n(\mathbb{R})$ of symmetric $n \times n$ matrices is a subring of $M_n(\mathbb{R})$, but *not* an ideal.
- The set $\mathbb{Z}$ is a subring of $\mathbb{Z}[x]$ but not an ideal.

Ideals

Remark

If an ideal I of R contains 1, then $I = R$.

Proof

Suppose $1 \in I$, and take an arbitrary $r \in R$.

Then $r1 \in I$, and so $r1 = r \in I$. Therefore, $I = R$. $\square$

We can modify the above result to show that if I contains *any* unit, then $I = R$. (HW)

Let's compare the concept of a normal subgroup to that of an ideal:

- normal subgroups are characterized by being invariant under conjugation:

$$H \leq G \text{ is normal} \quad \text{iff} \quad ghg^{-1} \in H \text{ for all } g \in G,\ h \in H.$$

- (left) ideals of rings are characterized by being invariant under (left) multiplication:

$$I \subseteq R \text{ is a (left) ideal} \quad \text{iff} \quad ri \in I \text{ for all } r \in R,\ i \in I.$$

Ideals generated by sets

Definition

The left ideal generated by a set $X \subset R$ is defined as:

$$(X) := \bigcap \{I : \; I \text{ is a left ideal s.t. } X \subseteq I \subseteq R\}.$$

This is the smallest left ideal containing X.

There are analogous definitions by replacing "left" with "right" or "two-sided".

Recall the two ways to define the subgroup $\langle X \rangle$ generated by a subset $X \subseteq G$:

- "*Bottom up*": As the set of all finite products of elements in X;
- "*Top down*": As the intersection of all subgroups containing X.

Proposition (HW)

Let R be a ring *with unity*. The (left, right, two-sided) ideal generated by $X \subseteq R$ is:

- Left: $\{r_1x_1 + \cdots + r_nx_n : n \in \mathbb{N},\, r_i \in R,\, x_i \in X\}$,
- Right: $\{x_1r_1 + \cdots + x_nr_n : n \in \mathbb{N},\, r_i \in R,\, x_i \in X\}$,
- Two-sided: $\{r_1x_1s_1 + \cdots + r_nx_ns_n : n \in \mathbb{N},\, r_i, s_i \in R,\, x_i \in X\}$.

Ideals generated by sets

As we did with groups, if $S = \{x\}$, we can write (x) rather than $(\{x\})$, etc.

Let's see some examples of ideals in $R = \mathbb{Z}[x]$.

$$(x) = \big\{xf(x) \mid f \in \mathbb{Z}[x]\big\} = \big\{a^n x^n + \cdots + a_1 x \mid a_i \in \mathbb{Z}\big\}.$$

$$(2) = \big\{2f(x) \mid f \in \mathbb{Z}[x]\big\} = \big\{2a^n x^n + \cdots + 2a_1 x + 2a_0 \mid a_i \in \mathbb{Z}\big\}.$$

$$(x, 2) = \big\{xf(x) + 2g(x) \mid f, g \in \mathbb{Z}[x]\big\} = \big\{a^n x^n + \cdots + a_1 x + 2a_0 \mid a_i \in \mathbb{Z}\big\}.$$

Notice that we have

$$(x) \subsetneq (x, 2) \subsetneq R, \qquad \text{and} \qquad (2) \subsetneq (x, 2) \subsetneq R.$$

The ideal $(x, 2)$ is said to be maximal, because there is nothing "between" it and R.

Question

How different would these ideals be in the ring $R = \mathbb{Q}[x]$?

Ideals and quotients

Since an ideal I of R is an additive subgroup (and hence normal), then:

- $R/I = \{x + I \mid x \in R\}$ is the set of cosets of I in R;
- R/I is a quotient group; with the binary operation (addition) defined as

$$(x + I) + (y + I) := x + y + I.$$

It turns out that if I is also a two-sided ideal, then we can make R/I into a ring.

Proposition

If $I \subseteq R$ is a (two-sided) ideal, then R/I is a ring (called a quotient ring), where multiplication is defined by

$$(x + I)(y + I) := xy + I\,.$$

Proof

We need to show this is well-defined. Suppose $x + I = r + I$ and $y + I = s + I$. This means that $x - r \in I$ and $y - s \in I$.

It suffices to show that $xy + I = rs + I$, or equivalently, $xy - rs \in I$:

$$xy - rs = xy - ry + ry - rs = (x - r)y + r(y - s) \in I\,.$$

Finite fields

We've already seen that:

- $\mathbb{Z}_p$ is a field if p is prime
- every finite integral domain is a field.

But *what do these "other" finite fields look like?*

Let $R = \mathbb{Z}_2[x]$ be the polynomial ring over $\mathbb{Z}_2$. (Note: we can ignore all negative signs.)

The polynomial $f(x) = x^2 + x + 1$ is irreducible over $\mathbb{Z}_2$ because it does not factor as a product $f(x) = g(x)h(x)$ of lower-degree terms. (Note that $f(0) = f(1) = 1 \neq 0$.)

Consider the ideal $I = (x^2 + x + 1)$, the set of multiples of $x^2 + x + 1$.

In the quotient ring R/I, we have the relation $x^2 + x + 1 = 0$, or equivalently,

$$x^2 = -x - 1 = x + 1.$$

The quotient has only 4 elements:

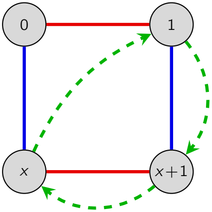
$$0 + I\,, \qquad 1 + I\,, \qquad x + I\,, \qquad (x + 1) + I.$$

As with the quotient group (or ring) $\mathbb{Z}/n\mathbb{Z}$, we usually drop the "I", and just write

$$R/I = \mathbb{Z}_2[x]/(x^2 + x + 1) \cong \{0,\ 1,\ x,\ x + 1\}.$$

Finite fields

Here are the Cayley diagram and Cayley tables for $R/I = \mathbb{Z}_2[x]/(x^2+x+1)$:



$+$	0	1	x	$x+1$
0	0	1	x	$x+1$
1	1	0	$x+1$	x
x	x	$x+1$	0	1
$x+1$	$x+1$	x	1	0

$\times$	1	x	$x+1$
1	1	x	$x+1$
x	x	$x+1$	1
$x+1$	$x+1$	1	x

Theorem

There exists a finite field $\mathbb{F}_q$ of order q, which is unique up to isomorphism, iff $q = p^n$ for some prime p. If $n > 1$, then this field is isomorphic to the quotient ring

$$\mathbb{Z}_p[x]/(f),$$

where f is *any* irreducible polynomial of degree n.

Much of the error correcting techniques in coding theory are built using mathematics over $\mathbb{F}_{2^8} = \mathbb{F}_{256}$. This is what allows DVDs to play despite scratches.

Motivation (spoilers!)

Many of the big ideas from group homomorphisms carry over to ring homomorphisms.

Group theory

- The quotient group G/N exists iff N is a normal subgroup.
- A homomorphism is a structure-preserving map: $f(x * y) = f(x) * f(y)$.
- The kernel of a homomorphism is a normal subgroup: $\text{Ker}(\phi) \trianglelefteq G$.
- For every normal subgroup $N \trianglelefteq G$, there is a natural quotient homomorphism $\phi\colon G \to G/N$, $\phi(g) = gN$.
- There are four standard isomorphism theorems for groups.

Ring theory

- The quotient ring R/I exists iff I is a two-sided ideal.
- A homomorphism is a structure-preserving map: $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$.
- The kernel of a homomorphism is a two-sided ideal: $\text{Ker}(\phi) \trianglelefteq R$.
- For every two-sided ideal $I \trianglelefteq R$, there is a natural quotient homomorphism $\phi\colon R \to R/I$, $\phi(r) = r + I$.
- There are four standard isomorphism theorems for rings.

Ring homomorphisms

Definition

A ring homomorphism is a function $f\colon R \to S$ satisfying

$$f(x+y) = f(x) + f(y) \qquad \text{and} \qquad f(xy) = f(x)f(y) \quad \text{for all } x, y \in R.$$

A ring isomorphism is a homomorphism that is bijective.

The kernel $f\colon R \to S$ is the set $\mathrm{Ker}(f) := \{x \in R \mid f(x) = 0\}$.

Examples

1. The ring homomorphism $\phi\colon \mathbb{Z} \to \mathbb{Z}_n$ sending $k \mapsto k \pmod{n}$ has $\mathrm{Ker}(\phi) = n\mathbb{Z}$.
2. For a fixed real number $\alpha \in \mathbb{R}$, the "evaluation function"

$$\phi\colon \mathbb{R}[x] \longrightarrow \mathbb{R}\,, \qquad \phi\colon p(x) \longmapsto p(\alpha)$$

   is a homomorphism. The kernel consists of all polynomials that have α as a root.
3. The following is a homomorphism, for the ideal $I = (x^2 + x + 1)$ in $\mathbb{Z}_2[x]$:

$$\phi\colon \mathbb{Z}_2[x] \longrightarrow \mathbb{Z}_2[x]/I, \qquad f(x) \longmapsto f(x) + I\,.$$

Ring homomorphisms

Proposition

The kernel of a ring homomorphism $\phi\colon R \to S$ is a two-sided ideal.

Proof

We know that $\mathrm{Ker}(\phi)$ is an additive subgroup of R.

We must show that it's a subring, and an ideal.

Subring: Let $k_1, k_2 \in \mathrm{Ker}(\phi)$. Then

$$\phi(k_1 k_2) = \phi(k_1)\phi(k_2) = 0 \cdot 0 = 0,$$

and so $k_1 k_2 \in \mathrm{Ker}(\phi)$. ✓

Left ideal: Let $k \in \mathrm{Ker}(\phi)$ and $r \in R$. Then

$$\phi(rk) = \phi(r)\phi(k) = r \cdot 0 = 0,$$

and so $rk \in \mathrm{Ker}(\phi)$. ✓

Showing that $\mathrm{Ker}(\phi)$ is a right ideal is analogous. □

The isomorphism theorems for rings

All of the isomorphism theorems for groups have analogues for rings.

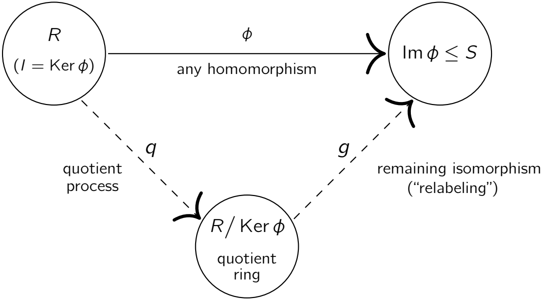
- Fundamental homomorphism theorem: "*All homomorphic images are quotients*"
- Correspondence theorem: Characterizes "*subrings and ideals of quotients*"
- Freshman theorem: Characterizes "*quotients of quotients*"
- Diamond isomorphism theorem: characterizes "*quotients of a sum*"

Since a ring is an abelian group with extra structure, we often don't have to prove these from scratch.

The FHT for rings: all homomorphic images are quotients

Fundamental homomorphism theorem for rings

If $\phi\colon R \to S$ is a ring homomorphism, then $\operatorname{Ker}\phi$ is an ideal and $\operatorname{Im}(\phi) \cong R/\operatorname{Ker}(\phi)$.



Proof (HW)

The statement holds for the underlying additive group R. Thus, it remains to show that $\operatorname{Ker}\phi$ is a (two-sided) ideal, and the following map is a ring homomorphism:

$$g\colon R/I \longrightarrow \operatorname{Im}\phi\,, \qquad\qquad g(x+I) = \phi(x)\,.$$

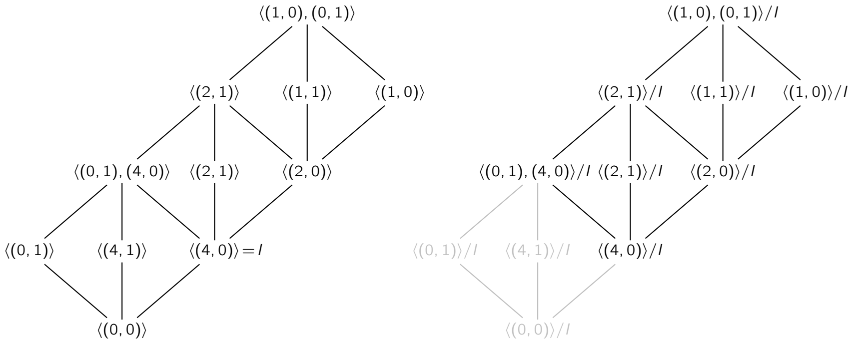
The correspondence theorem: subrings of quotients

Correspondence theorem

Let I be an ideal of R. There is a bijective correspondence between subrings of R/I and subrings of R that contain I.

Moreover every ideal of R/I has the form J/I, for some ideal J satisfying $I \subseteq J \subseteq R$.

Here is an example for the ring $R = \mathbb{Z}_8 \times \mathbb{Z}_2$:



Maximal ideals and simple rings

Define a maximal normal subgroup M of G is one for which there are no normal subgroups properly between them.

Formally, we can write this as

$$M \leq N \leq G, \quad \text{and} \quad M, N \trianglelefteq G \qquad \Longrightarrow \qquad N = M, \text{ or } N = G.$$

By the correspondence theorem, M is a maximal normal subgroup iff G/M is simple.

We can define analogous terms for rings.

Definition

A (proper) ideal I of R is maximal if $I \subseteq J \subseteq R$ holds implies $J = I$ or $J = R$.

A ring R is simple if its only (two-sided) ideals are 0 and R.

The following is immediate by the correspondence theorem.

Remark

An ideal M of R is maximal iff R/M is simple.

Maximal ideals and simple rings

Simple rings have no nontrivial proper ideals. Proper ideals cannot contain units.

In a field, *every* nonzero element is a unit. Therefore, fields have no nontrivial proper ideals.

Proposition

A commutative ring R is simple iff it is a field.

Proof

"$\Rightarrow$": Assume R is simple. Then $(a) = R$ for any nonzero $a \in R$.

Thus, $1 \in (a)$, so $1 = ba$ for some $b \in R$, so $a \in U(R)$ and R is a field. ✓

"$\Leftarrow$": Let $I \subseteq R$ be a nonzero ideal of a field R. Take any nonzero $a \in I$.

Then $a^{-1}a \in I$, and so $1 \in I$, which means $I = R$. ✓ $\square$

Theorem

Let R be a commutative ring with 1. The following are equivalent for an ideal $I \subseteq R$.

(i) I is a maximal ideal;

(ii) R/I is simple;

(iii) R/I is a field.

Maximal ideals

Recall that in a commutative ring, an ideal $M \neq 0$ is a maximal iff R/M is a field.

Let's see some examples.

1. The maximal ideals of $R = \mathbb{Z}$ are of the form $M = (p)$, where p is prime. The quotient field is $\mathbb{Z}/(p) \cong \mathbb{Z}_p$.

2. The maximal ideals of $R = \mathbb{Z}[x]$ are of the form

$$(x, p) = \big\{xf(x) + p \cdot g(x) \mid f, g \in \mathbb{Z}[x]\big\} = \big\{a^n x^n + \cdots + a_1 x + pa_0 \mid a_i \in \mathbb{Z}\big\}.$$

In the quotient field, "$x := 0$" and "$p := 0$", and so

$$\mathbb{Z}[x]/(x, p) = \big\{a_0 + M \mid a_0 = 0, \dots, p-1\big\} \cong \mathbb{Z}_p.$$

3. Let $R = \mathbb{Q}[x]$. The ideal

$$(x) = \big\{xf(x) \mid f \in \mathbb{Q}[x]\big\} = \big\{a^n x^n + \cdots + a_1 x \mid a_i \in \mathbb{Z}\big\}$$

is maximal. In the quotient field, "$x := 0$", and so

$$\mathbb{Q}[x]/(x) = \big\{a_0 + M \mid a_0 \in \mathbb{Q}\big\} \cong \mathbb{Q}.$$

4. Let $R = \mathbb{Z}_2[x]$. The ideal $M = (x^2 + x + 1)$ is maximal, and $R/M \cong \mathbb{F}_4$, the (unique) finite field of order 4.

Maximal ideals

The following is equivalent to the axiom of choice from set theory.

Zorn's lemma

If $\mathcal{P} \neq \emptyset$ is a poset in which every chain has an upper bound, then $\mathcal{P}$ has a maximal element.

Proposition

If R is a ring with 1, then every ideal $I \neq R$ is contained in a maximal ideal $M \lneq R$.

Proof

Let $\mathcal{P} = \{J \leq R \mid I \subseteq J \subsetneq R\}$, ordered by inclusion.

Every chain $\mathcal{C}$ has a maximal element, $L_{\mathcal{C}} = \displaystyle\bigcup_{J \in \mathcal{C}} J$, and hence an upper bound.

By Zorn's lemma, there is some maximal element M in $\mathcal{P}$, which is a maximal ideal.

The freshman theorem: quotients of quotients

The correspondence theorem characterizes the subring structure of the quotient R/J.

Every subring of R/I is of the form J/I, where $I \leq J \leq R$.

Moreover, if $J \trianglelefteq R$ is an ideal, then $J/I \trianglelefteq R/I$. In this case, we can ask:

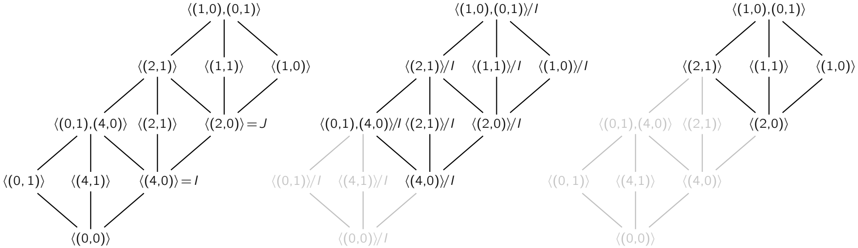
What is the quotient ring $(R/I)/(J/I)$ isomorphic to?

Freshman theorem

Suppose R is a ring with ideals $I \subseteq J$. Then J/I is an ideal of R/I and

$$(R/I)/(J/I) \cong R/J.$$

Here is an example for the ring $R = \mathbb{Z}_8 \times \mathbb{Z}_2$:



The freshman theorem: quotients of quotients

For another visualization, consider $R = \mathbb{Z}_6 \times \mathbb{Z}_4$ and write elements as strings.

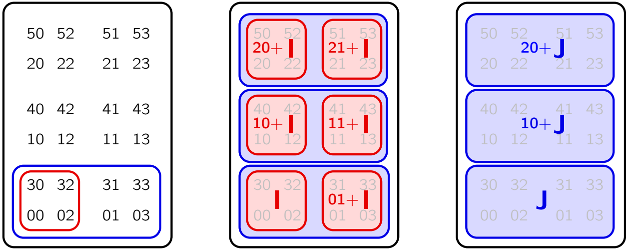
Consider the ideals $J = \langle 30, 02 \rangle \cong V_4$ and $I = \langle 30, 01 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_4$.

Notice that $I \leq J \leq R$, and $I = J \cup (01{+}J)$, and

$$R/I = \big\{I,\ 01{+}I,\ 10{+}I,\ 11{+}I,\ 20{+}I,\ 21{+}I\big\}, \qquad J/I = \{I,\ 01{+}I\}$$

$$R/J = \Big\{I \cup (01{+}I),\ (10{+}I) \cup (11{+}I),\ (20{+}I) \cup (21{+}I)\Big\}$$

$$(R/I)/(J/I) = \Big\{\{I,\ 01{+}I\},\ \{10{+}I,\ 11{+}I\},\ \{20{+}I,\ 21{+}I\}\Big\}.$$



$I \leq J \leq R$

R/I consists of 6 cosets
$J/I = \{I,\ 01{+}I\}$

R/J consists of 3 cosets
$(R/I)/(J/I) \cong R/J$

The diamond isomorphism theorem: quotients of sums

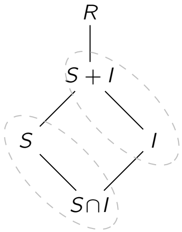
Diamond isomorphism theorem

Suppose S is a subring and I an ideal of R. Then

(i) The sum $S+I=\{s+i \mid s\in S,\ i\in I\}$ is a subring of R and the intersection $S\cap I$ is an ideal of S.

(ii) The following quotient rings are isomorphic:

$$(S+I)/I \cong S/(S\cap I)\,.$$



Proof (sketch)

$S+I$ is an additive subgroup, and it's closed under multiplication because

$$s_1, s_2 \in S,\ i_1, i_2 \in I \quad\Longrightarrow\quad (s_1+i_1)(s_2+i_2) = \underbrace{s_1 s_2}_{\in S} + \underbrace{s_1 i_2 + i_1 s_2 + i_1 i_2}_{\in I} \in S+I.$$

Showing $S\cap I$ is an ideal of S is straightforward (homework exercise).

We already know that $(S+I)/I \cong S/(S\cap I)$ as additive groups.

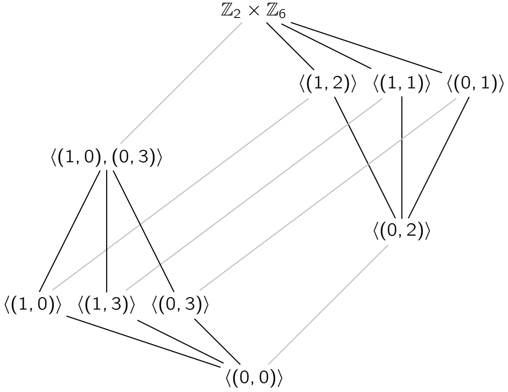
One explicit isomorphism is $\phi\colon s+(S\cap I)\mapsto s+I$. It is easy to check that $\phi\colon 1\mapsto 1$ and ϕ preserves products. $\square$

The diamond isomorphism theorem: quotients of products by factors

Let $R = \mathbb{Z}_2 \times \mathbb{Z}_6$, and consider the subring $S = \langle (1,0),(0,3) \rangle$ and ideal $I = \langle (0,2) \rangle$.

Then $R = I + J$, and $I \cap J = \langle (0,0) \rangle$.

Let's interpret the diamond theorem $(S + I)/I \cong S/S \cap I$ in terms of the subgroup lattice.



Prime ideals

Definition

Let R be a commutative ring. An ideal $P \subset R$ is prime if $ab \in P$ implies either $a \in P$ or $b \in P$.

Note that $p \in \mathbb{N}$ is a prime number iff $p = ab$ implies either $a = p$ or $b = p$.

Examples

1. The ideal (n) of $\mathbb{Z}$ is a prime ideal iff n is a prime number (possibly $n = 0$).
2. In the polynomial ring $\mathbb{Z}[x]$, the ideal $I = (2, x)$ is a prime ideal. It consists of all polynomials whose constant coefficient is even.

Theorem

An ideal $P \subseteq R$ is prime iff R/P is an integral domain.

The proof is straightforward (HW). Since fields are integral domains, the following is immediate:

Corollary

In a commutative ring, every maximal ideal is prime.

Divisibility and factorization

A ring is in some sense, a generalization of the familiar number systems like $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$, where we are allowed to add, subtract, and multiply.

Two key properties about these structures are:

- multiplication is commutative,
- there are no (nonzero) zero divisors.

Blanket assumption

Henceforth, unless explicitly mentioned otherwise, R is assumed to be an integral domain, and we will define $R^* := R \setminus \{0\}$.

The integers have several basic properties that we usually take for granted:

- every nonzero number can be factored uniquely into primes;
- any two numbers have a unique greatest common divisor and least common multiple;
- there is a Euclidean algorithm, which can find the gcd of two numbers.

Surprisingly, these need not always hold in integrals domains! We would like to understand this better.

Divisibility

Definition

If $a, b \in R$, say that a divides b, or b is a multiple of a if $b = ac$ for some $c \in R$. We write $a \mid b$.

If $a \mid b$ and $b \mid a$, then a and b are associates, written $a \sim b$.

Examples

- In $\mathbb{Z}$: n and $-n$ are associates.
- In $\mathbb{R}[x]$: $f(x)$ and $c \cdot f(x)$ are associates for any $c \neq 0$.
- The only associate of 0 is itself.
- The associates of 1 are the units of R.

Proposition (HW)

Two elements $a, b \in R$ are associates if and only if $a = bu$ for some unit $u \in U(R)$.

This defines an equivalence relation on R, and partitions R into equivalence classes.

Note that units divide everything: if $b \in R$ and $u \in U(R)$, then $u \mid b$.

Definition

If $b \notin U(R)$ and its only divisors are units and associates of b, then b is irreducible.

An element $p \in R$ is prime if p is not a unit, and $p \mid ab$ implies $p \mid a$ or $p \mid b$.

Proposition

If $0 \neq p \in R$ is prime, then p is irreducible.

Proof

Suppose p is not irreducible. Then $p = ab$ with $a, b \notin U(R)$.

Then (wlog) $p \mid a$, so $a = pc$ for some $c \in R$. Now,

$$p = ab = (pc)b = p(cb)\,.$$

This means that $cb = 1$, and thus $b \in U(R)$. Therefore, p is prime. $\square$

Irreducibles and primes

Caveat: Irreducible $\not\Rightarrow$ prime

Consider the ring $R_{-5} := \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\}$.

$$3 \mid (2+\sqrt{-5})(2-\sqrt{-5}) = 9 = 3 \cdot 3\,,$$

but $3 \nmid 2+\sqrt{-5}$ and $3 \nmid 2-\sqrt{-5}$.

Thus, 3 is irreducible in R_{-5} but *not* prime.

When irreducibles fail to be prime, we can lose nice properties like unique factorization.

Things can get really bad: not even the *lengths* of factorizations into irreducibles need be the same!

For example, consider the ring $R = \mathbb{Z}[x^2, x^3]$. Then

$$x^6 = x^2 \cdot x^2 \cdot x^2 = x^3 \cdot x^3.$$

The element $x^2 \in R$ is not prime because $x^2 \mid x^3 \cdot x^3$ yet $x^2 \nmid x^3$ in R (note: $x \notin R$).

Principal ideal domains

Fortunately, there is a type of ring where such "bad things" don't happen.

Definition

An ideal I generated by a single element $a \in R$ is called a principal ideal. We denote this by $I = (a)$.

If every ideal of R is principal, then R is a principal ideal domain (PID).

Examples

The following are all PIDs (stated without proof):

- The ring of integers, $\mathbb{Z}$.
- Any field F.
- The polynomial ring $F[x]$ over a field.

As we will see shortly, PIDs are "nice" rings. Here are some properties they enjoy:

- pairs of elements have a "greatest common divisor" & "least common multiple"
- irreducible $\Rightarrow$ prime
- Every element factors uniquely into primes.

Greatest common divisors & least common multiples

Proposition

If $I \subseteq \mathbb{Z}$ is an ideal, and $a \in I$ is its smallest positive element, then $I = (a)$.

Proof

Pick any positive $b \in I$. Write $b = aq + r$, for $q, r \in \mathbb{Z}$ and $0 \leq r < a$.

Then $r = b - aq \in I$, so $r = 0$. Therefore, $b = qa \in (a)$. $\square$

Definition

A common divisor of $a, b \in R$ is an element $d \in R$ such that $d \mid a$ and $d \mid b$.

Moreover, d is a greatest common divisor (GCD) if $c \mid d$ for all other common divisors c of a and b.

A common multiple of $a, b \in R$ is an element $m \in R$ such that $a \mid m$ and $b \mid m$.

It's a least common multiple (LCM) if $m \mid n$ for all other common multiples n of a and b.

Nice properties of PIDs

Proposition

If R is a PID, then any $a, b \in R^*$ have a GCD, $d = \gcd(a, b)$.

It is *unique up to associates*, and can be written as $d = xa + yb$ for some $x, y \in R$.

Proof

Existence. The ideal generated by a and b is

$$I = (a, b) = \{ua + vb : u, v \in R\}.$$

Since R is a PID, we can write $I = (d)$ for some $d \in I$, and so $d = xa + yb$.

Since $a, b \in (d)$, both $d \mid a$ and $d \mid b$ hold.

If c is a divisor of a & b, then $c \mid xa + yb = d$, so d is a GCD for a and b. ✓

Uniqueness. If d' is another GCD, then $d \mid d'$ and $d' \mid d$, so $d \sim d'$. ✓ □

Nice properties of PIDs

Corollary

If R is a PID, then every irreducible element is prime.

Proof

Let $p \in R$ be irreducible and suppose $p \mid ab$ for some $a, b \in R$.

If $p \nmid a$, then $\gcd(p, a) = 1$, so we may write $1 = xa + yp$ for some $x, y \in R$. Thus

$$b = (xa + yp)b = x(ab) + (yb)p\,.$$

Since $p \mid x(ab)$ and $p \mid (yb)p$, then $p \mid x(ab) + (yb)p = b$. $\square$

Not surprisingly, least common multiples also have a nice characterization in PIDs.

Proposition (HW)

If R is a PID, then any $a, b \in R^*$ have an LCM, $m = \operatorname{lcm}(a, b)$.

It is *unique up to associates*, and can be characterized as a generator of the ideal $I := (a) \cap (b)$.

Unique factorization domains

Definition

An integral domain is a unique factorization domain (UFD) if:

(i) Every nonzero element is a product of irreducible elements;

(ii) Every irreducible element is prime.

Examples

1. $\mathbb{Z}$ is a UFD: Every integer $n \in \mathbb{Z}$ can be uniquely factored as a product of irreducibles (primes):
$$n = p_1^{d_1} p_2^{d_2} \cdots p_k^{d_k}.$$
This is the *fundamental theorem of arithmetic*.

2. The ring $\mathbb{Z}[x]$ is a UFD, because every polynomial can be factored into irreducibles. But it is not a PID because the following ideal is not principal:
$$(2, x) = \{f(x) : \text{ the constant term is even}\}.$$

3. The ring R_{-5} is not a UFD because $9 = 3 \cdot 3 = (2 + \sqrt{-5})(2 - \sqrt{-5})$.

4. We've shown that (ii) holds for PIDs. Next, we will see that (i) holds as well.

Unique factorization domains

Theorem

If R is a PID, then R is a UFD.

Proof

We need to show Condition (i) holds: every element is a product of irreducibles. A ring is Noetherian if every ascending chain of ideals

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$

stabilizes, meaning that $I_k = I_{k+1} = I_{k+2} = \cdots$ holds for some k.

Suppose R is a PID. It is not hard to show that R is Noetherian (HW). Define

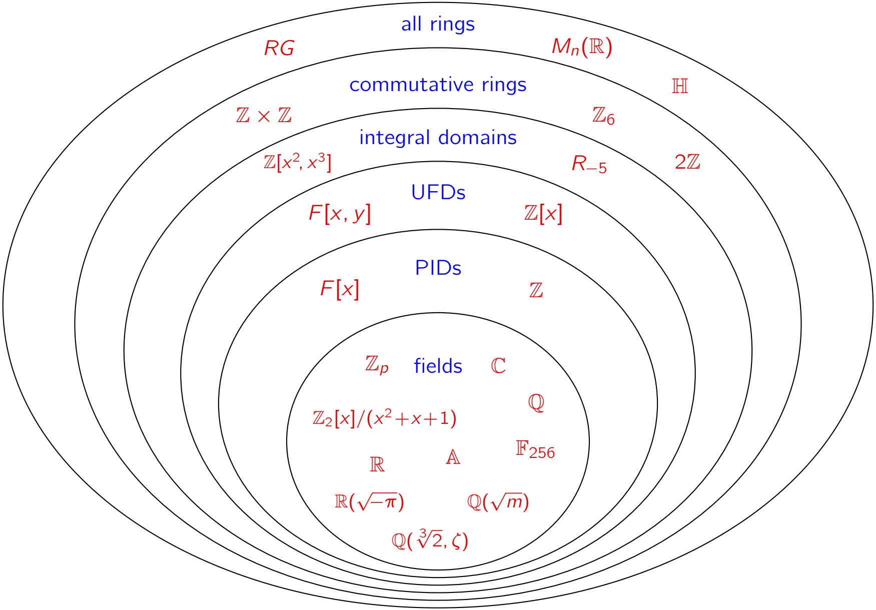
$$X = \{a \in R^* \setminus U(R) : a \text{ can't be written as a product of irreducibles}\}.$$

If $X \neq \emptyset$, then pick $a_1 \in X$. Factor this as $a_1 = a_2 b$, where $a_2 \in X$ and $b \notin U(R)$. Then $(a_1) \subsetneq (a_2) \subsetneq R$, and repeat this process. We get an ascending chain

$$(a_1) \subsetneq (a_2) \subsetneq (a_3) \subsetneq \cdots$$

that does not stabilize. This is impossible in a PID, so $X = \emptyset$. $\square$

Summary of ring types



The Euclidean algorithm

Around 300 B.C., Euclid wrote his famous book, the *Elements*, in which he described what is now known as the Euclidean algorithm:

Proposition VII.2 (Euclid's *Elements*)

Given two numbers not prime to one another, to find their greatest common measure.

The algorithm works due to two key observations:

- If $a \mid b$, then $\gcd(a, b) = a$;
- If $a = bq + r$, then $\gcd(a, b) = \gcd(b, r)$.

This is best seen by an example: Let $a = 654$ and $b = 360$.

$$\begin{aligned} 654 &= 360 \cdot 1 + 294 & \gcd(654, 360) &= \gcd(360, 294) \\ 360 &= 294 \cdot 1 + 66 & \gcd(360, 294) &= \gcd(294, 66) \\ 294 &= 66 \cdot 4 + 30 & \gcd(294, 66) &= \gcd(66, 30) \\ 66 &= 30 \cdot 2 + 6 & \gcd(66, 30) &= \gcd(30, 6) \\ 30 &= 6 \cdot 5 & \gcd(30, 6) &= 6. \end{aligned}$$

We conclude that $\gcd(654, 360) = 6$.

Euclidean domains

Loosely speaking, a Euclidean domain is a ring for which the Euclidean algorithm works.

Definition

An integral domain R is Euclidean if it has a degree function $d\colon R^* \to \mathbb{Z}$ satisfying:

(i) non-negativity: $d(r) \geq 0 \quad \forall r \in R^*$.

(ii) monotonicity: $d(a) \leq d(ab)$ for all $a, b \in R^*$.

(iii) division-with-remainder property: For all $a, b \in R$, $b \neq 0$, there are $q, r \in R$ such that

$$a = bq + r \qquad \text{with} \qquad r = 0 \quad \text{or} \quad d(r) < d(b)\,.$$

Note that Property (ii) could be restated to say: *If $a \mid b$, then $d(a) \leq d(b)$*;

Since 1 divides every $x \in R$,

$$d(1) \leq d(x), \qquad \text{for all } x \in R.$$

Similarly, if x divides 1, then $d(x) \leq d(1)$. Elements that divide 1 are the units of R.

Proposition

If u is a unit, then $d(u) = d(1)$. □

Euclidean domains

Examples

- $R = \mathbb{Z}$ is Euclidean. Define $d(r) = |r|$.
- $R = F[x]$ is Euclidean if F is a field. Define $d(f(x)) = \deg f(x)$.
- The Gaussian integers

$$R_{-1} = \mathbb{Z}[\sqrt{-1}] = \{a + bi \mid a, b \in \mathbb{Z}\}$$

is Euclidean with degree function $d(a + bi) = a^2 + b^2$.

Proposition

If R is Euclidean, then $U(R) = \{x \in R^* \mid d(x) = d(1)\}$.

Proof

We've already established "$\subseteq$". For "$\supseteq$", Suppose $x \in R^*$ and $d(x) = d(1)$.

Write $1 = qx + r$ for some $q \in R$, and $r = 0$ or $d(r) < d(x) = d(1)$.

But $d(r) < d(1)$ is impossible, and so $r = 0$, which means $qx = 1$ and hence $x \in U(R)$. $\square$

Euclidean domains

Proposition

If R is Euclidean, then R is a PID.

Proof

Let $I \neq 0$ be an ideal and pick some $b \in I$ with $d(b)$ minimal.

Pick $a \in I$, and write $a = bq + r$ with either $r = 0$, or $d(r) < d(b)$.

This latter case is impossible: $r = a - bq \in I$, and by minimality, $d(b) \leq d(r)$.

Therefore, $r = 0$, which means $a = bq \in (b)$. Since a was arbitrary, $I = (b)$. $\square$

Exercises.

(i) The ideal $I = (3, 2 + \sqrt{-5})$ is not principal in R_{-5}.

(ii) If R is an integral domain, then $I = (x, y)$ is not principal in $R[x, y]$.

Corollary

The rings R_{-5} (not a PID or UFD) and $R[x, y]$ (not a PID) are not Euclidean.

Algebraic integers

The algebraic integers are the roots of *monic* polynomials in $\mathbb{Z}[x]$. This is a subring of the algebraic numbers (roots of all polynomials in $\mathbb{Z}[x]$).

Assume $m \in \mathbb{Z}$ is square-free with $m \neq 0, 1$. Recall the quadratic field

$$\mathbb{Q}(\sqrt{m}) = \left\{ p + q\sqrt{m} \mid p, q \in \mathbb{Q} \right\}.$$

Definition

The ring R_m is the set of algebraic integers in $\mathbb{Q}(\sqrt{m})$, i.e., the subring consisting of those numbers that are roots of monic quadratic polynomials $x^2 + cx + d \in \mathbb{Z}[x]$.

Facts

- R_m is an integral domain with 1.
- Since m is square-free, $m \not\equiv 0 \pmod 4$. For the other three cases:

$$R_m = \begin{cases} \mathbb{Z}[\sqrt{m}] = \left\{ a + b\sqrt{m} : a, b \in \mathbb{Z} \right\} & m \equiv 2 \text{ or } 3 \pmod 4 \\ \mathbb{Z}\left[\frac{1+\sqrt{m}}{2}\right] = \left\{ a + b\left(\frac{1+\sqrt{m}}{2}\right) : a, b \in \mathbb{Z} \right\} & m \equiv 1 \pmod 4 \end{cases}$$

- R_{-1} is the Gaussian integers, which is a PID. (easy)
- R_{-19} is a PID. (hard)

Algebraic integers

Definition

For $x = r + s\sqrt{m} \in \mathbb{Q}(\sqrt{m})$, define the norm of x to be

$$N(x) = (r + s\sqrt{m})(r - s\sqrt{m}) = r^2 - ms^2 .$$

R_m is norm-Euclidean if it is a Euclidean domain with $d(x) = |N(x)|$.

Note that the norm is multiplicative: $N(xy) = N(x)N(y)$.

Exercises

Assume $m \in \mathbb{Z}$ is square-free, with $m \neq 0, 1$.

- $u \in U(R_m)$ iff $|N(u)| = 1$.
- If $m \geq 2$, then $U(R_m)$ is infinite.
- $U(R_{-1}) = \{\pm 1, \pm i\}$ and $U(R_{-3}) = \left\{ \pm 1, \pm \frac{1 \pm \sqrt{-3}}{2} \right\}$.
- If $m = -2$ or $m < -3$, then $U(R_m) = \{\pm 1\}$.

Euclidean domains and algebraic integers

Theorem

R_m is norm-Euclidean iff

$$m \in \{-11, -7, -3, -2, -1, 2, 3, 5, 6, 7, 11, 13, 17, 19, 21, 29, 33, 37, 41, 57, 73\}.$$

Theorem (D.A. Clark, 1994)

The ring R_{69} is a Euclidean domain that is *not* norm-Euclidean.

Let $\alpha = (1 + \sqrt{69})/2$ and $c > 25$ be an integer. Then the following degree function works for R_{69}, defined on the prime elements:

$$d(p) = \begin{cases} |N(p)| & \text{if } p \neq 10 + 3\alpha \\ c & \text{if } p = 10 + 3\alpha \end{cases}$$

Theorem

If $m < 0$ and $m \notin \{-11, -7, -3, -2, -1\}$, then R_m is not Euclidean.

Open problem

Classify which R_m's are PIDs, and which are Euclidean.

PIDs that are not Euclidean

Theorem

If $m < 0$, then R_m is a PID iff

$$m \in \{\underbrace{-1, -2, -3, -7, -11}_{\text{Euclidean}}, -19, -43, -67, -163\}.$$

Recall that R_m is norm-Euclidean iff

$$m \in \{-11, -7, -3, -2, -1, 2, 3, 5, 6, 7, 11, 13, 17, 19, 21, 29, 33, 37, 41, 57, 73\}.$$

Corollary

If $m < 0$, then R_m is a PID that is not Euclidean iff $m \in \{-19, -43, -67, -163\}$.

Algebraic integers

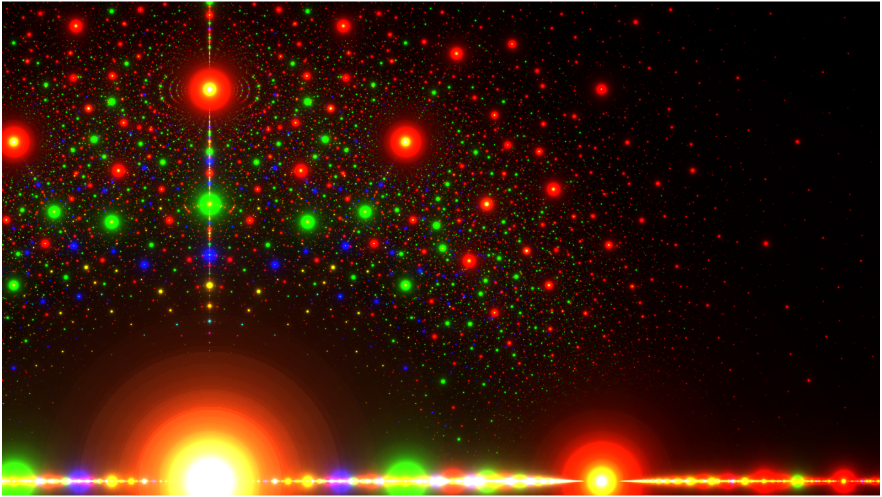


Figure: Algebraic numbers in the complex plane. Colors indicate the coefficient of the leading term: red = 1 (algebraic integer), green = 2, blue = 3, yellow = 4. Large dots mean fewer terms and smaller coefficients. Image from Wikipedia (made by Stephen J. Brooks).

Algebraic integers

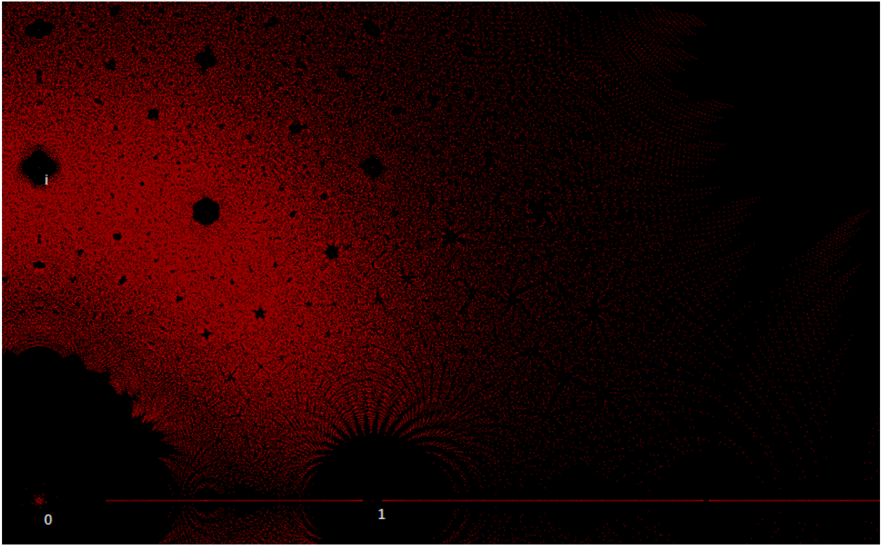
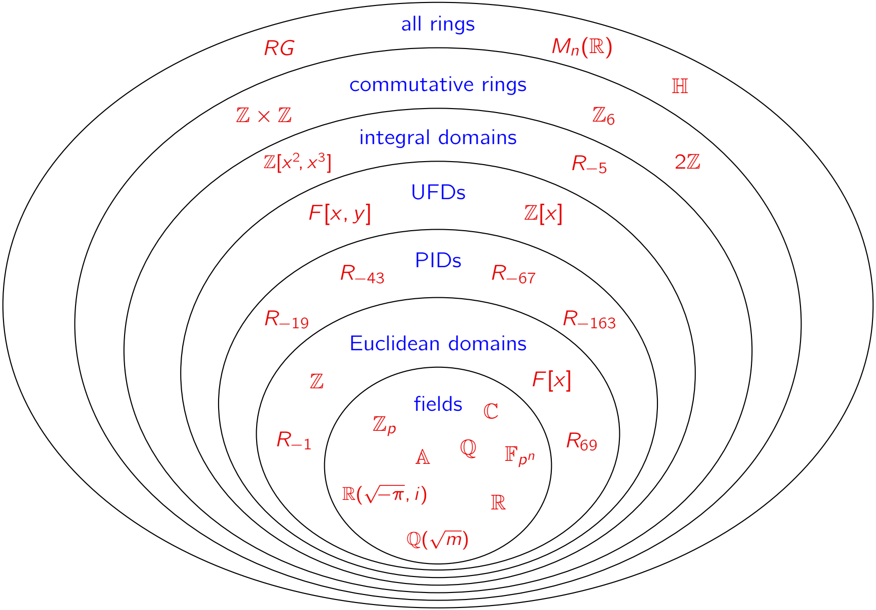


Figure: Algebraic integers in the complex plane. Each red dot is the root of a monic polynomial of degree ≤ 7 with coefficients from $\{0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5\}$. From Wikipedia.

Summary of ring types



Field of fractions

Rings allow us to add, subtract, and multiply, but not necessarily divide.

In any ring: if $a \in R$ is not a zero divisor, then $ax = ay$ implies $x = y$. *This holds even if a^{-1} doesn't exist.*

In other words, by allowing "divison" by non zero-divisors, we can think of R as a subring of a bigger ring that contains a^{-1}.

If $R = \mathbb{Z}$, then this construction yields the rational numbers, $\mathbb{Q}$.

If R is an integral domain, then this construction yields the field of fractions of R.

Goal

Given a commutative ring R, construct a larger ring in which $a \in R$ (that's not a zero divisor) has a multiplicative inverse.

Elements of this larger ring can be thought of as fractions. It will naturally contain an isomorphic copy of R as a subring:

$$R \hookrightarrow \left\{ \frac{r}{1} : r \in R \right\}.$$

From $\mathbb{Z}$ to $\mathbb{Q}$

Let's examine how one can construct the rationals from the integers.

There are many ways to write the same rational number, e.g., $\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \cdots$

Equivalence of fractions

Given $a, b, c, d \in \mathbb{Z}$, with $b, d \neq 0$,

$$\frac{a}{b} = \frac{c}{d} \qquad \text{if and only if} \qquad ad = bc.$$

Addition and multiplication is defined as

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \qquad \text{and} \qquad \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

It is not hard to show that these operations are well-defined.

The integers $\mathbb{Z}$ can be identified with the subring $\left\{\frac{a}{1} : a \in \mathbb{Z}\right\}$ of $\mathbb{Q}$, and every $a \neq 0$ has a multiplicative inverse in $\mathbb{Q}$.

We can do a similar construction in any commutative ring!

Rings of fractions

Blanket assumptions

- R is a commutative ring.
- $D \subseteq R$ is nonempty, multiplicatively closed $[d_1, d_2 \in D \Rightarrow d_1 d_2 \in D]$, and contains no zero divisors.
- Consider the following set of ordered pairs:

$$\mathcal{F} = \{(r, d) \mid r \in R,\ d \in D\},$$

Define an equivalence relation: $(r_1, d_1) \sim (r_2, d_2)$ iff $r_1 d_2 = r_2 d_1$. Denote this equvalence class containing (r_1, d_1) by $\frac{r_1}{d_1}$, or r_1/d_1.

Definition

The ring of fractions of D with respect to R is the set of equivalence classes, $R_D := \mathcal{F}/\sim$, where

$$\frac{r_1}{d_1} + \frac{r_2}{d_2} := \frac{r_1 d_2 + r_2 d_1}{d_1 d_2} \qquad \text{and} \qquad \frac{r_1}{d_1} \times \frac{r_2}{d_2} := \frac{r_1 r_2}{d_1 d_2}.$$

Rings of fractions

Basic properties (HW)

1. These operations on $R_D = \mathcal{F}/\sim$ are well-defined.
2. $(R_D, +)$ is an abelian group with identity $\frac{0}{d}$, for any $d \in D$. The additive inverse of $\frac{a}{d}$ is $\frac{-a}{d}$.
3. Multiplication is associative, distributive, and commutative.
4. R_D has multiplicative identity $\frac{d}{d}$, for any $d \in D$.

Examples

1. Let $R = \mathbb{Z}$ (or $R = 2\mathbb{Z}$) and $D = R - \{0\}$. Then the ring of fractions is $R_D = \mathbb{Q}$.
2. If R is an integral domain and $D = R - \{0\}$, then R_D is a field, called the field of fractions.
3. If $R = F[x]$ and $D = \{x^n \mid n \in \mathbb{Z}\}$, then $R_D = F[x, x^{-1}]$, the Laurent polynomials over F.
4. If $R = \mathbb{Z}$ and $D = 5\mathbb{Z}$, then $R_D = \mathbb{Z}[\frac{1}{5}]$, which are "polynomials in $\frac{1}{5}$" over $\mathbb{Z}$.
5. If R is an integral domain and $D = \{d\}$, then $R_D = R[\frac{1}{d}]$, the set of all "polynomials in $\frac{1}{d}$" over R.

Universal property of the ring of fractions

This says R_D is the "smallest" ring contaning R and all fractions of elements in D:

Theorem

Let S be any commutative ring with 1 and let $\varphi \colon R \hookrightarrow S$ be any ring embedding such that $\phi(d)$ is a unit in S for every $d \in D$.

Then there is a unique ring embedding $\Phi \colon R_D \to S$ such that $\Phi \circ q = \varphi$.

$$\begin{tikzcd} R \arrow[rr, "\varphi"] \arrow[dr, "q"'] & & S \\ & R/D \arrow[ur, dashed, "\Phi"'] & \end{tikzcd} \qquad \begin{tikzcd} r \arrow[rr, mapsto, "\varphi"] \arrow[dr, mapsto, "q"'] & & s \\ & r/1 \arrow[ur, mapsto, "\Phi"'] & \end{tikzcd}$$

Proof

Define $\Phi \colon R_D \to S$ by $\Phi(r/d) = \varphi(r)\varphi(d)^{-1}$. This is well-defined and 1–1. (HW)

Uniqueness. Suppose $\Psi \colon R_D \to S$ is another embedding with $\Psi \circ q = \varphi$. Then

$$\Psi(r/d) = \Psi((r/1) \cdot (d/1)^{-1}) = \Psi(r/1) \cdot \Psi(d/1)^{-1} = \varphi(r)\varphi(d)^{-1} = \Phi(r/d).$$

Thus, $\Psi = \Phi$. $\square$