

Read: Lax, Chapter 1, pages 1–11.

1. Let X be a vector space over a field K . Let 0 be the zero element of K and $\mathbf{0}$ the zero-element of X . Using only the definitions of a group, a vector space, and a field, carefully prove each of the following:
 - (a) The identity element e of a group is unique.
 - (b) In any group G , the inverse of $g \in G$ is unique.
 - (c) $0x = \mathbf{0}$ for every $x \in X$;
 - (d) $k\mathbf{0} = \mathbf{0}$ for every $k \in K$;
 - (e) For every $k \in K$ and $x \in X$, if $kx = \mathbf{0}$, then $k = 0$ or $x = \mathbf{0}$.
2. Let X be a vector space over K , and let S be a linearly independent subset of X . The following is called the *exchange property*: For any nonzero $x_0 \in \text{Span}(S)$, there exists $x_1 \in S$ such that $S' = (S \setminus \{x_1\}) \cup \{x_0\}$ is a basis for $\text{Span}(S)$.
 - (a) Prove the exchange property.
 - (b) Suppose that B is a size- n basis for X , and let B' be another basis. Use the exchange property to show that $|B'| = n$.
3. Let X_1, X_2 be a subspace of a finite-dimensional K -vector space X . Show that $\dim(X_1 \times X_2) = \dim(X_1 \oplus X_2)$.
4. If Y is a subspace of X , then two vectors $x_1, x_2 \in X$ are *congruent modulo Y* , denoted $x_1 \equiv x_2 \pmod{Y}$, if $x_1 - x_2 \in Y$. This is an equivalence relation; denote the equivalence class containing $x \in X$ by $\{x\}$, and let X/Y denote the set of equivalence classes. We can make X/Y into a vector space by defining addition and scalar multiplication as follows:

$$\{x\} + \{z\} := \{x + z\}, \quad a\{x\} := \{ax\}.$$

- (a) Show that these operations are *well-defined*. That is, they do not depend on the choice of congruence class representatives.
- (b) Prove that for any $x \in X$, the following two sets are equal as subsets of X :

$$\{x\} = \{x + y \mid y \in Y\}.$$

This motivates the alternative “coset notation” of $x + Y$ for the equivalence class of x modulo Y . Show how to add and scalar multiply in this notation by computing:

$$(x + Y) + (z + Y), \quad \text{and} \quad a(x + Y), \quad x, z \in X, a \in K.$$

- (c) Assume $\dim X < \infty$. Show that X is isomorphic to $Y \times X/Y$ by defining an explicit map and showing that it is linear and a bijection.
5. Let Z be a subspace of X .
 - (a) Show that if $Z \leq Y \leq X$, then Y/Z is a subspace of X/Z .
 - (b) Show that *every* subspace of X/Z arises in this manner.