

Read: Lax, Chapter 2, pages 13–18.

1. Let  $K$  be a finite field. The *characteristic* of  $K$ , denoted  $\text{char } K$ , is the smallest positive integer  $n$  for which  $n1 := \underbrace{1 + 1 + \cdots + 1}_{n \text{ times}} = 0$ .
  - (a) Prove that the characteristic of  $K$  is prime.
  - (b) Show that  $K$  is a vector space over  $\mathbb{Z}_p$ , where  $p = \text{char } K$ .
  - (c) Show that  $|K|$  is a prime power.
  - (d) Show that if  $K$  and  $L$  are finite fields with  $K \subset L$  and  $|K| = p^m$  and  $|L| = p^n$ , then  $m$  divides  $n$ .
2. Let  $X$  be a vector space over a field  $K$  and let  $X'$  be the set of linear functions from  $X$  to  $K$ , also known as the *dual space* of  $X$ .
  - (a) Let  $x_1, \dots, x_n$  be a basis for  $X$ . Define  $\ell_j \in X'$  by  $\ell_j(x_i) = \delta_{ij}$ . Show that  $\ell_1, \dots, \ell_n$  is a basis for  $X'$ ; it is called the *dual basis* of  $x_1, \dots, x_n$ .
  - (b) Find the dual basis of  $x_1 = (1, -1, 3)$ ,  $x_2 = (0, 1, -1)$ , and  $x_3 = (0, 3, -2)$  in  $X = \mathbb{R}^3$ .
  - (c) Express the scalar function  $f \in X'$ , where  $f(x, y, z) = 2x - y + 3z$  as a linear combination of the dual basis,  $\ell_1, \ell_2, \ell_3$ , from Part (b).

3. Let  $S$  be a subset of  $X$ . The *annihilator* of  $S$  is the set

$$S^\perp = \{\ell \in X' \mid \ell(s) = 0 \text{ for all } s \in S\}.$$

- (a) Show that  $\text{span}(S)$  is the intersection of all subspaces  $T_\alpha$  of  $X$  that contain  $S$ :

$$\text{span}(S) = \bigcap_{S \subseteq T_\alpha \subseteq X} T_\alpha,$$

making it well-founded to speak of the “*smallest subspace of  $X$  that contains  $S$ .*”

- (b) Show that  $S^\perp$  is a subspace of  $X'$ , and that  $S^\perp = \text{span}(S)^\perp$ .

4. Let  $\mathcal{P}_2$  be the vector space of all polynomials  $p(x) = a_0 + a_1x + a_2x^2$  over  $\mathbb{R}$ , with degree  $\leq 2$ . Let  $\xi_1, \xi_2, \xi_3$  be distinct real numbers, and define

$$\ell_j: \mathcal{P}_2 \longrightarrow \mathbb{R}, \quad \ell_j(p) = p(\xi_j) \quad \text{for } j = 1, 2, 3.$$

- (a) Show that  $\ell_1, \ell_2, \ell_3$  is a basis for the dual space  $\mathcal{P}'_2$ .
- (b) Find polynomials  $p_1(x), p_2(x), p_3(x)$  in  $\mathcal{P}_2$  of which  $\ell_1, \ell_2, \ell_3$  is the dual basis in  $\mathcal{P}'_2$ .

5. Let  $W$  be the subspace of  $\mathbb{R}^4$  spanned by  $(1, 0, -1, 2)$  and  $(2, 3, 1, 1)$ . Find the annihilator of  $W$ . Write your answer by giving an explicit basis of  $W^\perp$  in terms of the dual basis vectors  $\ell_1, \dots, \ell_4$ .