

Read: Lax, Chapter 3, pages 19–25, 28–31.

1. Let  $T: X \rightarrow U$  be a linear map. Prove the following:
  - (a) The image of a subspace of  $X$  is a subspace of  $U$ .
  - (b) The inverse image of a subspace of  $U$  is a subspace of  $X$ .
  
2. Let  $X$  and  $U$  be vector spaces, and suppose that  $Y$  is a subspace of  $X$ . Let  $Q: X \rightarrow X/Y$  be the canonical quotient map sending  $x \mapsto \{x\}$ , and let  $T: X \rightarrow U$  be a linear map. State and prove a condition that is both necessary and sufficient for the existence of a unique linear map  $S: X/Y \rightarrow U$  such that  $T = S \circ Q$ . When this happens, the map  $T$  is said to *factor through* the quotient space, as shown by the following commutative diagram:

$$\begin{array}{ccc}
 X & \xrightarrow{U} & U \\
 \searrow Q & & \nearrow S \\
 & X/Y &
 \end{array}$$

3. Suppose  $T: X \rightarrow X$  is a linear map of rank 1.
  - (a) Show that there exists  $c \in K$  such that  $T^2 = cT$ .
  - (b) Show that if  $c \neq 1$ , then  $I - T$  has an inverse.
  
4. Suppose that  $S, T: X \rightarrow X$  are linear maps.
  - (a) Show that  $\text{rank}(S + T) \leq \text{rank}(S) + \text{rank}(T)$ .
  - (b) Show that  $\text{rank}(ST) \leq \text{rank}(S)$ .
  - (c) Show that  $\dim(N_{ST}) \leq \dim N_S + \dim N_T$ .

For each of these, without using matrices, give an explicit example of where equality holds, and another where it does not.

5. Let  $T: X \rightarrow X$  be linear, with  $\dim X = n$ .
  - (a) Prove that if  $T^2 = T$ , then  $X = R_T \oplus N_T$ .
  - (b) Show by example that if  $T^2 \neq T$ , then  $X = R_T \oplus N_T$  need not hold.
  - (c) Prove that  $N_{T^n} = N_{T^{n+1}}$  and  $R_{T^n} = R_{T^{n+1}}$ .
  - (d) Prove that  $X = R_{T^n} \oplus N_{T^n}$ .
  - (e) Show there exists a linear map  $S: X \rightarrow X$  such that  $ST = TS$  and  $ST^{n+1} = T^n$ .