Read: Lax, Chapter 5, pages 44–54.

1. Let S_n denote the set of all permutations of $\{1, \dots, n\}$.
   (a) Prove that $\operatorname{sgn}(\pi_1 \circ \pi_2) = \operatorname{sgn}(\pi_1)\operatorname{sgn}(\pi_2)$.
   (b) Prove that $\operatorname{sgn}(\tau) = -1$ for all transpositions $\tau \in S_n$.
   (c) Let $\pi \in S_n$, and suppose that $\pi = \tau_k \circ \cdots \circ \tau_1 = \sigma_\ell \circ \cdots \circ \sigma_1$, where $\tau_i, \sigma_j \in S_n$ are transpositions. Prove that $k \equiv \ell \bmod 2$.

2. Let f be a *non-degenerate* symmetric bilinear form over an n-dimensional vector space X. That is, for all nonzero $x \in X$, there is some $y \in X$ for which $f(x, y) \neq 0$. Consequently, fixing any nonzero $x \in X$ defines a nonzero dual vector

$$f(x, -) \in X', \qquad f(x, -) \colon y \longmapsto f(x, y).$$

   (a) Prove that the map $L_f \colon X \to X'$ given by $L_f \colon x \mapsto f(x, -)$ is an isomorphism.
   (b) Let $x_1, \dots, x_n$ be a basis for X. Express the dual basis $\ell_1, \dots, \ell_n$ in this form. That is, find g for which $L_g \colon x_i \mapsto \ell_i$.
   (c) Show how to construct another basis $y_1, \dots, y_n$ such that $f(x_i, y_j) = \delta_{ij}$.
   (d) Conversely, prove that if $\mathcal{B}_X = \{x_1, \dots, x_n\}$ and $\mathcal{B}_Y = \{y_1, \dots, y_n\}$ are sets of vectors in X with $f(x_i, y_j) = \delta_{ij}$, then $\mathcal{B}_X$ and $\mathcal{B}_Y$ are bases for X.

3. Let f be a non-degenerate symmetric bilinear form over a real n-dimensional vector space, where $1 + 1 \neq 0$.
   (a) Show that there exists $x_1 \in X$ with $f(x_1, x_1) \neq 0$.
   (b) Let Z_1 be the nullspace of $f(x_1, -)$. Show that f restricted to Z_1 is non-degenerate.
   (c) Construct a basis $\{z_1, \dots, z_n\}$ for X that satisfies $f(z_i, z_j) = \delta_{ij}$.

4. Let f be a bilinear form over a vector space X with basis $\{x_1, x_2\}$.
   (a) Assume f is alternating. Determine a formula for $f(u, v)$ in terms of each $f(x_i, x_j)$ and the coefficients used to express u and v with this basis. [Pun intented!]
   (b) Repeat Part (a) but assume that f is symmetric and $f(x, x) = 0$ for all $x \in X$.

5. Let X be an n-dimensional vector space over a field K.
   (a) Show how any bilinear form can be expressed as a sum of a symmetric and a skew-symmetric one. Describe the differences in the cases of $\operatorname{char} K \neq 2$ and $\operatorname{char} K = 2$.
   (b) Give an example of a non-alternating skew-symmetric mulitlinear form.
   (c) Give an example of a non-zero alternating multilinear form such that $f(x_1, \dots, x_k) = 0$ for some set of linearly independent vectors $x_1, \dots, x_k$.