Read: Lax, Chapter 6, pages 58–76.

1. Let $A\colon X \to X$ be a linear map with distinct eigenvalues $\lambda_1, \dots, \lambda_n$ and corresponding eigenvectors $v_1, \dots, v_n$. Let $\ell_1, \dots, \ell_n$ be the dual basis.

   (a) Prove that $\ell_1, \dots, \ell_n$ are eigenvectors of the transpose map $A'\colon X' \to X'$.

   (b) Now, suppose that $f_1, \dots, f_n$ is *any* basis of eigenvectors of A'. Prove that $(f_i, v_j) = 0$ if $i \neq j$ and $(f_i, v_i) \neq 0$.

   (c) For any $x = a_1 v_1 + \cdots + a_n v_n$, derive a formula for a_i in terms of x, v_i, and f_i.

2. Let A be an invertible $n \times n$ matrix. Prove that A^{-1} can be written as a polynomial in degree at most $n-1$. That is, prove that there are scalars c_i such that

$$A^{-1} = c_{n-1}A^{n-1} + c_{n-2}A^{n-2} + \cdots + c_1 A + c_0 I.$$

3. Let λ be an eigenvalue of A, and let N_j be the nullspace of $(A - \lambda I)^j$. Elements of N_j are called *generalized eigenvectors* of λ. The special case of $j = 1$ are the ordinary ("genuine") eigenvectors. Prove that $A - \lambda I$ extends to a well-defined map $N_{j+1}/N_j \longrightarrow N_j/N_{j-1}$, and that this mapping is 1–1.

4. Let A be an $n \times n$ matrix over $\mathbb{C}$ with an eigenvalue λ and corrresponding eigenvector v_1. Let v_2 be a generalized eigenvector satisfying $(A - \lambda I)v_2 = v_1$.

   (a) Show that $A^N v_2 = \lambda^N v_2 + N\lambda^{N-1} v_1$, for any $N \in \mathbb{N}$.

   (b) Show that $q(A)v_2 = q(\lambda)v_2 + q'(\lambda)v_1$, for any polynomial $q(t) \in \mathbb{C}[t]$.

   (c) Give a formula (no proof needed) for $q(A)v_m$, where $v_1, \dots, v_m$ are generalized eigenvectors of A with $(A - \lambda I)v_k = v_{k-1}$. Let $v_0 = 0$, for convenience.

5. Do the following for the matrix A below, and then repeat it for B:

$$A = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 1 \\ 0 & 0 & -1 & 0 \\ 4 & 0 & -6 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & -4 \\ 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad J_\lambda = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}.$$

   (a) Find the characteristic and minimal polynomials, and all (genuine) eigenvectors.

   (b) For each eigenvalue λ, compute $\dim N_{(A-\lambda I)^j}$ for $j = 1, 2, 3, \dots$.

   (c) Find a basis $\mathcal{B}$ of $\mathbb{C}^4$ consisting of generalized eigenvectors, so that the matrix with respect to this basis is $J = P^{-1}AP$, where J is a *Jordan matrix*. This means that J is block-diagonal formed from *Jordan blocks* J_λ; see above.

   (d) A subspace $Y \subseteq \mathbb{C}^4$ is A-*invariant* if $A(Y) \subseteq Y$. Of the 16 subspaces spanned by subsets of $\mathcal{B}$, determine which ones are A-invariant.