Read: Lax, Chapter 7, pages 89–100.

1. Let x_1, x_2 be a basis of $X = \mathbb{R}^2$, and ℓ_1, ℓ_2 the dual basis. Carry out the steps below for the linear map $A\colon X \to X$ defined by $A(x_1) = x_1$ and $A(x_2) = x_1 + x_2$ with respect to the standard dot product, and then with respect to each of the following inner products:
$$\langle x, y\rangle := \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \text{and} \qquad \langle x, y\rangle := \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$
   (a) Find $v_i \in X$ for which $\ell_i = \langle -, v_i\rangle$, for $i = 1, 2$.
   (b) Find $y_i \in X$ for which $A'(\ell_i) = \ell_i \circ A = \langle -, y_i\rangle$, for $i = 1, 2$.
   (c) Find the adjoint $A^*\colon X \to X$ with respect to this inner product.

2. Let $A\colon X \to U$ be a linear map between finite-dimensional inner product spaces, and let $A^*\colon U \to X$ denote the adjoint. Prove each of the following equalities:
   (a) $N_{A^*} = R_A^\perp$
   (b) $R_{A^*} = N_A^\perp$
   (c) $N_A = R_{A^*}^\perp$
   (d) $R_A = N_{A^*}^\perp$.

3. Let $A\colon X \to U$ be a linear map between finite-dimensional inner product spaces. The map A has a *left inverse* if there is a linear map $L\colon U \to X$ such that $LA = I_X$, the identity on X. It has a *right inverse* if there is a linear map $R\colon U \to X$ such that $AR = I_U$ is the identity on U.
   (a) Prove that A maps R_{A^*} bijectively onto R_A.
   (b) Show that if A has a left inverse, then $Ax = u$ has *at most* one solution. Give a condition on u that completely characterizes when there is a solution.
   (c) Show that if A has a right inverse, then $Ax = u$ has *at least* one solution. If $Ax_p = u$ for some particular $x_p \in X$, then describe all solutions for x in this case. What condition ensures that there will be *only* one solution?
   (d) What are the possibilities for the rank of A if it has a left inverse? What if it has a right inverse?

4. A projection $P\colon X \to X$ is a linear map such that $P^2 = P$.
   (a) Show that if P is a projection, then $X = R_P \oplus N_P$.
   (b) Show that $R_P^\perp = N_P$ if and only if P is self-adjoint.

5. Consider four data points $(0, 0)$, $(1, 8)$, $(3, 8)$, and $(4, 20)$. In this problem, we will find the best fit line $C + Dt$ through these points, using least squares.
   (a) Write down an equation in matrix form, $Ax = b$, where $x = (C, D)^T$, that has no solution because these four points are not co-linear.
   (b) Find the best fit line by solving the related equation $A\hat{x} = p$, where p is the orthogonal projection of b onto the range of A.
   (c) Repeat the previous steps to find the best fit parabola $C + Dt + Et^2$.