Read: Lax, Chapter 7, pages 89–100.

- 1. Let X be a complex inner product space, and $A: X \to X$.
 - (a) Show that if $\langle Ax, x \rangle = 0$ for all $x \in X$, then A = 0.
 - (b) Give an explicit example of how the previous part fails if X is a real inner product space.
 - (c) Show that A if self-adjoint if and only if $\langle Ax, x \rangle \in \mathbb{R}$ for all $x \in X$.
 - (d) Show that on a real inner product space, A is self-adjoint and $\langle Ax, x \rangle = 0$ for all $x \in X$, then A = 0.
- 2. Let X be the space of continuous complex-valued functions on [-1, 1] and define an inner product on X by

$$(f,g) = \int_{-1}^{1} f(s)\overline{g(s)} \, ds \, .$$

Let m(s) be a continuous function of absolute value 1, that is, $|m(s)| = 1, -1 \le s \le 1$. Define M to be multiplication by m:

$$(Mf)(s) = m(s)f(s).$$

Show that M is unitary.

- 3. Fix an orthonormal basis of \mathbb{C}^n , and let S be the cyclic shift mapping $S(a_1,\ldots,a_n)=(a_2,\ldots,a_n,a_1)$.
 - (a) Prove that S is unitary.
 - (b) Find the characteristic and minimal polynomials, eigenvalues, and eigenvectors of S.
 - (c) Find an orthonormal basis of \mathbb{C}^n consisting of eigenvectors of S.
- 4. Consider the quadratic form $q(x) = 2x_1^2 + 6x_1x_2 + 2x_2^2$.
 - (a) Write this as $q(x) = x^T A x$, for some A.
 - (b) Write $A = PDP^T$, where D is a diagonal matrix and P is orthogonal with determinant 1.
 - (c) Change variables by letting $z = P^T x$. Sketch the level curve q(x) = 1 in both the $z_1 z_2$ -plane and in the $x_1 x_2$ -plane.
- 5. Let $N: X \to X$ be a linear map of a finite-dimensional complex inner product space. Prove that the following are equivalent:
 - (i) N is normal (that is, $NN^* = N^*N$).
 - (ii) N is unitarily similar to a diagonal matrix (i.e., $N = UDU^*$).
 - (iii) Every eigenvector of N is an eigenvector of N^* .