

Read: Lax, Chapter 8, pages 101–120.

1. Let $N: X \rightarrow X$ be a normal mapping of an inner product space.
 - (a) Prove that $\|N\| = \max |n_i|$, where the n_i s are the eigenvalues of N .
 - (b) Show that N has a square-root, that is, a matrix R such that $N = R^2$. Is R necessarily normal? Unique?
2. Let $H: X \rightarrow X$ be self-adjoint, with eigenvalues $\lambda_1 \geq \dots \geq \lambda_n$. Prove the following *max-min* principle:

$$\lambda_k = \max_{\dim S=k} \min_{x \in S \setminus \{0\}} R_H(x).$$

3. For any positive mapping $M: X \rightarrow X$, define an inner product on X by $\langle x, y \rangle := (x, My)$. Throughout this problem, assume that $X = \mathbb{R}^2$ and $M = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$.
 - (a) Find two orthonormal bases for X that contain the vector $e_1/\|e_1\|$, where $e_1 = (1, 0)$.
 - (b) Find two orthonormal bases for X that contain the vector $e_2/\|e_2\|$, where $e_2 = (0, 1)$.
 - (c) Find an vector v_2 orthogonal to $v_1 = (1, 1)$.
 - (d) Find a matrix H that is self-adjoint with respect to $(\ , \)$, but *not* with respect to $\langle \ , \ \rangle$.

4. Let $H, M: X \rightarrow X$ be self-adjoint mappings, and M positive.
 - (a) Formulate and prove a necessary and sufficient condition for $M^{-1}H$ to be self-adjoint with respect to the standard inner product.
 - (b) Prove that $M^{-1}H$ is self-adjoint with respect to the inner product $\langle x, y \rangle = (x, My)$. Conclude that there exists a basis v_1, \dots, v_n of X and $\mu_1, \dots, \mu_n \in \mathbb{R}$ such that

$$Hv_i = \mu_i Mv_i, \quad \langle v_i, v_j \rangle = \delta_{ij}.$$

- (c) Find formulas for $\langle v_i, v_j \rangle$ and $\langle v_i, M^{-1}Hv_j \rangle$ in terms of the standard inner product.
 - (d) Show that if H has only positive eigenvalues, then so does $M^{-1}H$.
5. Let $H, M: X \rightarrow X$ be self-adjoint mappings, M positive, and define $R_{H,M}(x) = \frac{(x, Hx)}{(x, Mx)}$.

- (a) Show that $\mu_1 := \min\{R_{H,M}(x) \mid x \in X\}$ exists, and write an equation relating v_1, μ_1, H , and M .
 - (b) Show that there is some $v_2 \in X$ solving the constrained minimum problem

$$\mu_2 := \min \{R_{H,M}(x) \mid (x, Mv_1) = 0\}.$$

Write an equation relating v_2, μ_2, H , and M .

- (c) Find an invertible U and diagonal D such that $U^*MU = I$ and $U^*HU = D$.
 - (d) Characterize the diagonal entries of D by a min-max principle.