1. Consider the following matrices:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}, \qquad A^TA = \begin{bmatrix} 5 & 15 \\ 15 & 45 \end{bmatrix}, \qquad AA^T = \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}.$$

   (a) Find the eigenvalues σ_1^2, σ_2^2 and unit eigenvectors v_1, v_2 of A^TA.

   (b) For the $\sigma_i \neq 0$, compute $u_i = Av_i/\sigma_i$ and verify that indeed $||u_i|| = 1$. Find the other u_i by computing the other unit eigenvector of AA^T.

   (c) Construct the left polar decomposition, $A = UP$.

   (d) Construct the singular value decomposition (SVD), $A = U\Sigma V^T$.

   (e) Write down orthonormal bases for each the "four fundamental subspaces": the row space R_A, the nullspace N_A, the column space R_{A^T}, and the left nullspace N_{A^T}.

   (f) Describe *all* matrices that have the same four fundamental subspaces.

   (g) Find a left inverse, right inverse, and pseudoinverse of A, or explain why it doesn't exist.

2. Compute the polar and singular value decomposition of the rotation matrix

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

   where $a, b \in \mathbb{R}$.

3. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}.$$

   (a) Construct the singular value decomposition of A.

   (b) Write down orthonormal bases for each the "four fundamental subspaces": the row space R_A, the nullspace N_A, the column space R_{A^T}, and the left nullspace N_{A^T}.

   (c) Find a left inverse, right inverse, and pseudoinverse of A, or explain why it doesn't exist.