Lecture 1.2: Spanning, independence, and bases

Matthew Macauley

Department of Mathematical Sciences
Clemson University
http://www.math.clemson.edu/~macaule/

Math 8530, Advanced Linear Algebra

Overview

We previously introduced the notion of a vector space, which consists of:

- a set X of vectors
- a set $\mathbb{F}$ of scalars

such that X is closed under addition, subtraction, and scalar multiplication, and equipped with the natural associative and distributive laws.

We saw how linear maps are structure-preserving functions between vector spaces.

Finally, we learned about special subsets that are also vectors spaces, called subspaces.

In this lecture, we will look at subsets that are not necessarily subspaces, and learn what it means for them to be:

- spanning ("generates X")
- linearly independent ("no redundancies")
- a basis ("large enough to generate, but small enough to not be redundant")

We will also formalize what the dimension of a vector space is.

Spanning and independence

Definition

A linear combination of vectors $x_1, \ldots, x_k$ is a vector of the form $a_1x_1 + \cdots + a_kx_k$, where each $a_i \in K$.

Definition

Given a subset $S \subseteq X$, the subspace spanned by S is the set of all linear combinations of vectors in S, and denoted $\text{Span}(S)$.

Exercise

For any subset $S \subseteq X$,

$$\text{Span}(S) = \bigcap_{S \subseteq Y_\alpha \leq X} Y_\alpha,$$

where the intersection is taken over all subspaces of X that contain X. □

Definition

The vectors $x_1, \ldots, x_k$ are linearly dependent if we can write $a_1x_1 + \cdots + a_kx_k = 0$, where not all $a_i = 0$. Otherwise, the vectors are linearly independent.

Spanning vs. linear independence

Lemma 1.1

If $X = \text{Span}(x_1, \dots, x_n)$, and the vectors $y_1, \dots, y_k \in X$ are linearly independent, then $k \leq n$.

Proof

Basis of a vector space

Definition

A set $B \subseteq X$ is a basis for X if:

- B spans X. (is "big enough");
- B is linearly independent. (isn't "too big").

Exercise

The following are equivalent for a subset $B \subseteq X$:

(i) B is a basis of X;

(ii) B is a minimal spanning set;

(iii) B is a maximal linearly independent set.

Examples

Let's find bases for some familiar vector spaces.

1. $K^n = \{(a_1, \dots, a_n) : a_i \in K\}$. Addition and multiplication are defined componentwise.
2. Set of functions $S \longrightarrow K$ from a finite set S.
3. Set of polynomials of degree $< n$, with coefficients from K.

Bases

Lemma 1.2

If $\text{Span}(x_1, \dots, x_n) = X$, then some subset of $\{x_1, \dots, x_n\}$ is a basis for X.

Proof

Definition

A vector space X is finite dimensional (f.d.) if it has a finite basis.

Examples in $\mathbb{R}^n$

(i) One vector is linearly independent iff it is nonzero.

(ii) Two vectors are linearly independent iff they do not lie on the same line (i.e., aren't scalar multiples).

(iii) Three vectors are linearly independent iff they do not lie on the same plane.

Dimension

Theorem / Definition 1.3

All bases for a f.d. vector space have the same cardinality, called the dimension of X.

Proof

Theorem 1.4

Every linear independent set of vectors $y_1, \ldots, y_k$ in a finite-dimensional vector space X can be extended to a basis of X.

Proof

An example from ODEs

Let X be the set of all smooth functions $x(t)$ that satisfy the second order differential equation $\frac{d^2}{dt^2}x + x = 0$.

If $x_1(t)$, $x_2(t)$ are solutions, then so are $x_1(t) + x_2(t)$ and $cx_1(t)$. Thus X is a vector space.

Solutions describe the motion of a mass-spring system (simple harmonic motion). A particular solution is determined completely by specifying:

$$x(0) = x_0 \quad \text{(initial position)} \qquad\qquad x'(0) = v_0 \quad \text{(initial velocity)}.$$

Thus, we can describe an element $x(t) \in X$ by a pair (x_0, v_0), where $x_0, v_0 \in \mathbb{R}$ (or in $\mathbb{C}$).

This defines an isomorphism $X \longrightarrow \mathbb{C}^2$, by $x(t) \longmapsto (x(0), x'(0))$.

Note that $\cos x$ and $\sin x$ are two linearly independent solutions, so the general solution to this ODE is $a\cos x + b\sin x$; $a, b \in \mathbb{C}$.

Said differently, $\{\cos x, \sin x\}$ is a basis for the solution space of $x'' + x = 0$.

Note that $\cos x + i\sin x = e^{ix}$ and $\cos x - i\sin x = e^{-ix}$ are linearly independent, and so $\{e^{ix}, e^{-ix}\}$ is another basis! Thus, the general solution can be written as $C_1e^{ix} + C_2e^{-ix}$ instead!