Lecture 1.3: Direct products and sums

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Math 8530, Advanced Linear Algebra

Overview

In previous lectures, we learned about vectors spaces and subspaces.

We learned about what it meant for a subset to span, to be linearly independent, and to be a basis.

In this lecture, we will see how to create new vector spaces from old ones.

We will see several ways to "multiply" vector spaces together, and will learn how to construct:

- the complement of a subspace
- the direct sum of two subspaces
- the direct product of two vector spaces

Complements and direct sums

Theorem 1.5

(a) Every subspace Y of a finite-dimensional vector space X is finite-dimensional.

(b) Every subspace Y has a complement in X: another subspace Z such that every vector $x \in X$ can be written uniquely as

$$x = y + z, \qquad y \in Y,\ z \in Z, \qquad \dim X = \dim Y + \dim Z.$$

Proof

Definition

X is the direct sum of subspaces Y and Z that are complements of each other.

More generally, X is the direct sum of subspaces $Y_1, \dots, Y_m$ if every $x \in X$ can be expressed uniquely as

$$x = y_1 + \cdots + y_m, \qquad y_i \in Y_i.$$

We denote this as $X = Y_1 \oplus \cdots \oplus Y_m$.

Direct products

Definition

The direct product of X_1 and X_2 is the vector space

$$X_1 \times X_2 := \big\{(x_1, x_2) \mid x_1 \in X_1,\ x_2 \in X_2\big\},$$

with addition and multiplication defined component-wise.

Proposition

- $\dim(Y_1 \oplus \cdots \oplus Y_m) = \sum_{i=1}^m \dim Y_i$;
- $\dim(X_1 \times \cdots \times X_m) = \sum_{i=1}^m \dim X_i$.

Example

Let $X = \mathbb{R}^4, \quad Y_1 = \{(a, b, 0, 0) \mid a, b \in \mathbb{R}\}, \quad Y_2 = \{(0, 0, c, d) \mid c, d \in \mathbb{R}\}, \quad X_1 = X_2 = \mathbb{R}^2.$

Clearly, $X = Y_1 \oplus Y_2$, since $(a, b, c, d) = (a, b, 0, 0) + (0, 0, c, d)$ [uniquely].

$$X_1 \times X_2 = \Big\{\big((a, b), (c, d)\big) \mid (a, b) \in \mathbb{R}^2, (c, d) \in \mathbb{R}^2\Big\} \cong \big\{(a, b, c, d) \mid a, b, c, d \in \mathbb{R}\big\} = X.$$

Direct sums vs. direct products

In the finite-dimensional cases, there is no difference (up to isomorphism) of direct sums vs. direct products.

Not the case when $\dim X = \infty$. Consider the vector space:

$$X = \mathbb{R}^\infty := \{(a_1, a_2, a_3, \dots) \mid a_i \in \mathbb{R}\} \cong \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \cdots$$

and the following subspaces:

$$X_1 = \{(a_1, 0, 0, 0, \dots,) \mid a_1 \in \mathbb{R}\}, \qquad X_2 = \{(0, a_2, 0, 0, \dots,) \mid a_2 \in \mathbb{R}\}, \qquad \text{and so on.}$$

Elements in the subspace $X_1 \oplus X_2 \oplus X_3 \oplus \cdots$ of X are finite sums

$$x = x_{i_1} + x_{i_2} + \cdots + x_{i_k}, \qquad x_{i_j} \in X_{i_j}.$$

Thus, we can write the direct sum as follows:

$$X_1 \oplus X_2 \oplus X_3 \oplus \cdots = \{(a_1, \dots, a_k, 0, 0, \dots) \mid a_i \in \mathbb{R},\ k \in \mathbb{Z}\} \subsetneq \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \cdots$$

- Elements in the direct product are sequences, e.g., $x = (1, 1, 1, \dots)$.
- Elements in the direct sum are finite sums, e.g., $x = 3e_1 - 5.25e_4 + 78e_{11}$.